CATEGORICAL TORELLI THEOREM FOR NODAL FANO THREEFOLDS AND PARAMETRIZATION VIA BRIDGRLAND STABLE OBJECTS

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Abstract. Let $\mathcal{B}$ be a dg category, we define intermediate Jacobian for this dg category, generalizing the construction of Alexander Perry for the admissible subcategory of $D^b(X)$ for smooth projective variety X in [Per22, Definition 5.24]. As an application, we prove

(1) Certain nodal curve is determined by their bounded derived category.
(2) 1-nodal maximally non-factorial index one prime Fano threefolds of genus $g \geq 6$ is determined by its Kuznetsov component up to birational isomorphisms.
(3) A general genus 7 1-nodal maximally non-factorial prime Fano threefold is determined by its Kuznetsov component up to isomorphisms.

Then we show that 1-nodal maximally non-factorial index one prime Fano threefolds of genus $g \geq 6$ coming from bridge construction is determined by a smooth and proper subcategory of the Kuznetsov component together with a distinguished object. As an application, We describe the fiber of (categorical) period map for one nodal maximally non-factorial prime Fano threefolds of genus $g \geq 6$ from *bridge construction* via Bridgeland stable objects in Kuznetsov components of corresponding del Pezzo threefolds.

Contents

1. Introduction

The bounded derived category of coherent sheaves $\mathrm{D}^b(X)$ on a smooth Fano variety X contains the same information as the variety itself (cf. [BO01]). In the past thirty years, characterizing an algebraic variety and extracting its geometric properties such as rationality and birationality from its derived category has gradually become one of the main subjects in modern birational geometry. Among others, Kuznetsov launched the program of Homological projective duality (HPD)

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which offers a very important tool to study the bounded derived category of a projective variety together with its linear section, providing many interesting semi-orthogonal decompositions and equivalences between them. Probably the most influential conjecture in this area is a homological criterion of the rationality of cubic fourfolds ([Kuz10]) in terms of derived category of K3 surface. It has now been widely accepted that the Kuznetsov component–the non-trivial semi-orthogonal component which is usually defined as the orthogonal complement of an exceptional collection–of a smooth projective variety X encodes essential birational geometric information. A comprehensive study on the Kuznetsov components of a series smooth projective varieties was made, see [Kuz04b, Kuz16, Kuz09, KP18, BF11, BF13, BF14]. In particular, a notable conjecture was proposed in [Kuz09] by relating the Kuznetsov components of two species birationally equivalent (or conjectured to be birationally equivalent) prime Fano threefolds, which we recall below, Denote by $\mathcal{MF}^i_d$ the moduli space of smooth Fano threefold of index i and degree d.

Conjecture 1.1. *[Kuz09, Conjecture 3.7] There is a correspondence $\mathcal{Z}_d \subset \mathcal{MF}^2_d \times \mathcal{MF}^i_{4d+2}$, such that for any pair $(Y_d, X_{4d+2}) \in \mathcal{Z}_d$, there is an equivalence of categories*[1]

$$\mathcal{K}u(Y_d) \simeq \mathcal{K}u(X_{4d+2}).$$

There are many interesting geometric and Hodge theoretical consequences of Conjecture 1.1. One of the geometric consequences is the identification of Fano surface of lines on degree $d(3 \leq d \leq 5)$ Del Pezzo threefold with Fano surface of conics on prime Fano threefold of degree $4d+2$ [KPS18, Proposition B.4.1,B.5.1,B.6.1]. More generally, the Bridgeland moduli spaces of semistable objects with specific Mukai vectors in Kuznetsov components of these Fano threefolds are isomorphic [LZ21, Theorem 1.1]. Besides the results on moduli spaces, the authors of [FLZ23] identify the group of automorphisms of index one genus 8 prime Fano threefold with that of correspondent Phaffian cubic threefold. Hodge theoretically, the author in [Per22] introduces intermediate Jacobian $\mathrm{J}(\mathcal{A})$ of an admissible subcategory $\mathcal{A}$ of bounded derived category $D^b(X)$ of a smooth projective variety X. He shows that the intermediate Jacobian $\mathrm{J}(\mathcal{K}u(X))$ of the Kuznetsov component of a smooth Fano threefold X recovers classical intermediate Jacobian $J(X)$. In particular, the intermediate Jacobian of prime Fano threefold of degree $4d+2$ is isomorphic to that of Del Pezzo threefold of degree d for $d \geq 3$. As a result, the Kuznetsov component of these prime Fano threefolds serves as their birational invariant.

However, it turns out that Conjecture 1.1 is false for $d = 1$ and $d = 2$ by [Zha20], [BP23] and [LZ23] independently. To remedy this situation and correct the Conjecture 1.1, the authors in [KS22] and [KS23] study slightly degenerations of smooth prime Fano threefolds of index one, called 1-nodal maximally non-factorial Fano threefolds, which are constructed from smooth Del Pezzo threefold of Picard rank one via *bridge construction* in [CKGS23]. They introduce the notion of *categorical absorption of singularities* to relate the Kuznetsov component of nodal prime Fano threefolds to that of smooth Del Pezzo threefold as follows.

Theorem 1.2. *[KS23, Proposition 3.3] Let X be the 1-nodal Fano threefold constructed via the* bridge construction *in [CKGS23] from a smooth Del Pezzo threefold Y. Then there is a semi-orthogonal decomposition*

$$D^b(X) = \langle \mathcal{P}_X, \widetilde{\mathcal{A}_X}, \mathcal{O}_X, \mathcal{U}_X^\vee \rangle,$$

where the category $\mathcal{P}_X$ is responsible for 1-nodal singularity, which is generated by a $\mathbb{P}^{\infty,1}$-object(Definition 4.4), and $\widetilde{\mathcal{A}_X}$ is a smooth and proper category, moreover $\widetilde{\mathcal{A}_X} \simeq \mathcal{K}u(Y)$.

[1]The Kuznetsov's Fano threefold conjecture stated slightly different for it required the correspondence $\mathcal{Z}_d$ is dominant on both factors. By [BT16], the dimension counting rules out the possibility of dominance on the second factor, so it left open a question whether such an correspondence could exist.

In this context, we still define the Kuznetsov component $\mathcal{K}u(X)$ of these 1-nodal Fano threefold X as the orthogonal complement of the exceptional pair $\langle \mathcal{O}_X, \mathcal{U}_X^\vee \rangle$. As in the smooth case, we have the following natural question.

Question 1.3. How do we reconstruct intermediate Jacobians of these nodal varieties from their derived categories?

To answer this question, we study Blanc's topological K-theory for the category $\mathcal{K}u(X)$ (cf. [Bla16]) and associated Hodge structure. For this, we apply the method of noncommutative Hodge theory [Per22],[JLLZ23] to category $\mathcal{K}u(X)$, extending their results to arbitrary dg categories. More precisely, for an arbitrary dg category $\mathcal{B}$, we define *intermediate Jacobian*

$$J(\mathcal{B}) = \frac{HP_1(\mathcal{B})}{j(HN_{-1}(\mathcal{B})) + \mathrm{Im}\mathrm{Ch}\, K_1^{top}(\mathcal{B})},$$

where $\mathrm{HP}_1(\mathcal{B}), \mathrm{HN}_{-1}(B)$ are periodic cyclic homology and negative cyclic homology of $\mathcal{B}$ and $j : \mathrm{HN}_{-1}(\mathcal{B}) \to \mathrm{HP}_1(\mathcal{B})$ is the natural map. It is a priori an abelian group. We apply this construction to the *homological finite subcategory* $\mathcal{K}u(X)_{\mathrm{hf}}$ (see Definition 2.9) of the Kuznetsov component of 1-nodal maximally non-factorial Fano threefolds, recovering their intermediate Jacobians.

Theorem 1.4. *Let X be a 1-nodal maximally non-factorial prime Fano threefold of index one or index two. Then*

$$\mathrm{J}(\mathcal{K}u(X)_{\mathrm{hf}}) \cong J(X).$$

In particular, if X is an index 1-nodal prime Fano threefold of degree $4d+2$, then

$$\mathrm{J}(\mathcal{K}u(X_{4d+2})_{\mathrm{hf}}) \cong J(Y_d).$$

We say that *categorical Torelli theorem* holds for a Fano variety X if the isomorphism class of X is determined by its Kuznetsov component, while *birational categorical Torelli theorem* holds if $\mathcal{K}u(X)$ only determine its birational isomorphism class. Motivated by these statements, we apply Theorem 1.4 to 1-nodal maximally non-factorial prime Fano threefold of genus $g \geq 6$ and nodal curves to prove that the Kuznetsov components determine their (birational) isomorphism classes.

Theorem 1.5.

(1) *Let $C_i = C_i' \bigcup C_i''$ be a reducible Gorenstein curve with $C_i' \cong \mathbb{P}^1$ and C_i'' a smooth curve of genus $g(C_i'') > 1$ and $C_i' \bigcap C_i''$ is a single point x. If $D^b(C_1) \simeq D^b(C_2)$, then $C_1 \cong C_2$.*

(2) *Let X, X' be 1-nodal maximally non-factorial index one prime Fano threefolds of genus $g \geq 6$ such that $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$, then $X \simeq X'$.*

(3) *Let X and X' be 1-nodal maximally non-factorial genus 7 Fano threefolds such that both of them are constructed via the bridge construction from a genus 7 degree 8 curve C, which is general in the tetragonal locus. Then $\mathcal{K}u(X) \simeq \mathcal{K}u(X') \Longrightarrow X \cong X'$.*

To prove Theorem 1.5 for nodal Fano threefolds, we first show the Fourier-Mukai type equivalence $\Phi : \mathcal{K}u(X) \simeq \mathcal{K}u(X')$ would induce an isomorphism $\phi : J(X) \cong J(X')$ as principally polarized abelian varieties. Then using [KS23, Prop A.16], the isomorphism ϕ agrees with the isomorphism $\psi : J(Y) \cong J(Y')$ for corresponding smooth Del Pezzo threefold Y and Y'. Finally, the classical Torelli theorem for these Del Pezzo threefolds implies $Y \cong Y'$ and we obtain $X \simeq X'$. The proof for nodal curves is similar.

As an interesting application, we prove a birational reconstruction theorem of Bondal-Orlov style for these nodal Fano threefolds.

Theorem 1.6. *Let X, X' be 1-nodal maximally non-factorial index one prime Fano threefolds of genus $g \geq 6$ such that $D^b(X) \simeq D^b(X')$, then $X \simeq X'$.*

Then we consider the problem of determining the isomorphism class of a nodal prime Fano threefold X from its birational isomorphism class through an additional distinguished object in its Kuznetsov component, known as *Refined categorical Torelli problem.* In smooth cases, such an object is produced from the tautological quotient bundle, and refined categorical Torelli theorems are established in [JLZ22, Theorem 1.3]. For a maximally non-factorial nodal Fano threefold X, let

$$\begin{array}{ccc} & \widetilde{Y} & \\ \sigma \swarrow & & \searrow \pi \\ Y & & X \end{array}$$

be the *bridge construction* for X. According to [KS23, Proposition 3.3], the semi-orthogonal decomposition of $\widetilde{Y}$ is given by

$$D^b(\widetilde{Y}) = \langle \mathcal{O}_{\widetilde{Y}}(E-H), \mathbf{T}_{\mathcal{O}_{\widetilde{L}}(-1)}(\mathcal{O}_{\widetilde{Y}}(E-H)), \mathbf{R}_{\mathbf{T}_{\mathcal{O}_{\widetilde{L}(-1)}}(\mathcal{O}_{\widetilde{Y}}(E-H))}(\widetilde{\mathcal{B}}_Y), \mathcal{O}_{\widetilde{Y}}, \mathcal{U}^{\vee}_{\widetilde{Y}} \rangle,$$

where $\mathbf{T}_{\mathcal{O}_{\widetilde{L}}(-1)}(\mathcal{O}_{\widetilde{Y}}(E-H))$ is the spherical twist of $\mathcal{O}_{\widetilde{Y}}(E-H)$ via $\mathcal{O}_{\widetilde{L}}(-1)$. We still have the semi-orthogonal decomposition

$$D^b(X) = \langle \mathcal{K}u(X), \mathcal{O}_X, \mathcal{U}^{\vee}_X \rangle,$$

where $\mathcal{U}_X = \pi_* \mathcal{U}_{\widetilde{Y}}$, the Mukai bundle. Denote by $\mathcal{Q}^{\vee}_X := \mathbf{L}_{\mathcal{O}_X} \mathcal{U}^{\vee}_X[-1]$, which is a vector bundle if $d = H^3_Y \geq 3$. Denote by $i : \mathcal{K}u(X) \hookrightarrow \langle \mathcal{K}u(X), \mathcal{Q}^{\vee}_X \rangle$ the inclusion functor. Then it is natural to ask the following question.

Question 1.7. Let X be a 1-nodal maximally non-factorial prime Fano threefold of genus $g = 2d+2, d \geq 2$. Is the isomorphism class of X determined by $\mathcal{K}u(X)$ and $i^! \mathcal{Q}^{\vee}_X$?

In the article, we discuss the question for degree $2d+2, d \geq 2$, we show the category of absorption of singularities $\widetilde{\mathcal{A}_X} \subset \mathcal{K}u(X)$ together with the distinguished object $\mathbf{R}_{\mathcal{P}_X} i^! \mathcal{Q}^{\vee}_X \in \widetilde{\mathcal{A}_X}$ recover the exact data to reconstruct X. More precisely, we prove the following theorem.

Theorem 1.8. *Let X be a 1-nodal maximally non-factorial prime Fano threefold of genus $g \geq 6$, obtained from the* bridge construction. *Then the isomorphic class of X is determined by $\widetilde{\mathcal{A}_X} \subset \mathcal{K}u(X)$ and $\mathbf{R}_{\mathcal{P}_X} i^! \mathcal{Q}^{\vee}_X$, where $\mathcal{K}u(X) = \langle \mathcal{P}, \widetilde{\mathcal{A}_X} \rangle$. More precisely,*

(1) If $g = 6$, then the distinguished object $\sigma_* \circ \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} \circ \mathbf{L}_{\mathbf{T}} \circ \pi^*(i^! \mathcal{Q}^{\vee}_X) \cong (\sigma_* \circ \mathbf{L}_{\mathbf{T}}) \pi^*(\mathbf{R}_{\mathcal{P}_X} i^! \mathcal{Q}^{\vee}_X)$ is isomorphic to the ideal sheaf I_L of a line L we start with.

(2) If $g \geq 8$, then the distinguished object $\sigma_* \circ \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} \circ \mathbf{L}_{\mathbf{T}} \circ \pi^*(i^! \mathcal{Q}^{\vee}_X) \in \mathcal{K}u(Y)$ is isomorphic to the (acyclic extension) of non-locally free instanton sheaf associated to the smooth rational curve C of degree $d-1$ we start with.

In other words, *refined categorical Torelli theorem* is proved.

Theorem 1.9. *Let X, X' be 1-nodal Fano threefolds above and $\Phi : \widetilde{\mathcal{A}_X} \simeq \widetilde{\mathcal{A}_{X'}}$ be the equivalence such that $\Phi(\mathbf{R}_{\mathcal{P}} i^! \mathcal{Q}^{\vee}_X) \cong \mathbf{R}_{\mathcal{P}'} i^! \mathcal{Q}^{\vee}_{X'}$, then $X \cong X'$.*

1.1. **Fiber of categorical period map.** The idea of the categorical period map is introduced in [JLLZ21a, Remark 10.2] but was only rigorously defined in a recent paper [KS23, Section 1.7]. Denote by $\mathcal{S} \subset \overline{\mathrm{MFM}}_{X_g}$ the substack of the moduli stack of Fano-Mukai pairs defined in [KS23, Definition 1.3]. Let

$$\mathcal{P}_{\widetilde{A}} \colon \mathcal{S} \to \mathrm{MTrCat},$$

be the categorical period map, which sends the pair $(X, \mathcal{U}_X)$ of index one prime Fano threefold X of genus g and the Mukai bundle $\mathcal{U}_X$ to a smooth and proper semi-orthogonal component

$\widetilde{A_X}$(cf. Section 8). A natural question is to describe the fiber of the categorical period map $\mathcal{P}_{\widetilde{\mathcal{A}}}$. A beautiful application of the *refined categorical Torelli theorem* is identifying this fiber as a sub locus of the Bridgeland moduli space of semistable objects in the Kuznetsov component of some Fano threefold–a successful attempt is made in [JLZ22, Theorem 1.4] to compute the fiber of categorical period map for smooth Fano threefolds. We apply a similar idea, using Theorem 1.9 to compute the fiber of $\mathcal{P}_{\widetilde{\mathcal{A}}}$ for nodal Fano threefolds.

Theorem 1.10.

(1) *The fiber of categorical period map*

$$\mathcal{P}_{\widetilde{\mathcal{A}}}\colon \overline{\mathrm{MFM}_{\mathrm{X_6,Y_2}}} \to \mathrm{MTrCat}$$

- *over $\widetilde{\mathcal{A}_X}$ of a smooth a smooth Gushel-Mukai threefold X (which is nothing but Kuznetsov component $\mathcal{K}u(X)$) is the disjoint union of two surfaces: $\widetilde{Y}^{\geq 2}_{A^\perp} \bigcup \widetilde{Y}^{\geq 2}_{A}$.*
- *over $\widetilde{\mathcal{A}_X}$, where X is a 1-nodal maximally non-factorial Gushel-Mukai threefold via* bridge construction *from a quartic double solid Y is Hilbert scheme $F(Y)$ of lines on Y*

(2) *The fiber of categorical period map*

$$\mathcal{P}_{\widetilde{\mathcal{A}}}\colon \overline{\mathrm{MFM}_{\mathrm{X_8,Y_3}}} \to \mathrm{MTrCat}$$

over $\widetilde{\mathcal{A}_X}$ where X is either a smooth index one prime Fano threefold of genus 8 or a 1-nodal maximally non-factorial one via bridge construction *from a cubic threefold Y is isomorphic to the complement of the strictly semistable objects in moduli space $\mathcal{M}_\sigma(\mathcal{K}u(Y), 2[I_l])$ of semistable objects, which consists of rank two instanton bundles and rank two stable but non locally free instanton sheaves.*

(3) *For $d \geq 4$, the fiber of categorical period map*

$$\mathcal{P}_{\widetilde{\mathcal{A}}}\colon \overline{\mathrm{MFM}_{\mathrm{X_{2d+2},Y_d}}}^{(1)} \to \mathrm{MTrCat}$$

over $\widetilde{\mathcal{A}_X}$ with X to be a 1-nodal maximally non-factorial genus $2d+2$ Fano as above is the locus of acyclic extension of a rank two charge $d-1$ non locally free stable instanton sheaves F_0 of the form

$$0 \to F_0 \to \mathcal{O}_Y^{\oplus 2} \to \mathcal{O}_C((d-2)p) \to 0.$$

Inspired by Theorem above we make the following conjecture.

Conjecture 1.11. *Let*

$$\mathcal{P}_{\widetilde{\mathcal{A}}}\colon \overline{\mathrm{MFM}_{X_{2d+2}}}^{\leq 1\mathrm{mnf}} \to \mathrm{MTrCat},$$

with $d \geq 3$ be the categorical period map on moduli stack consisting of smooth component MFM_{X_g} *and 1-nodal maximally non-factorial component of Fano-Mukai pairs.*

Then for any point $\widetilde{\mathcal{A}_X}$ in its image, the fiber

$$\mathcal{P}^{-1}_{\widetilde{\mathcal{A}}}(\widetilde{\mathcal{A}_X}) \cong \mathcal{M}_\sigma(\mathcal{K}u(Y_d), (d-1)[I_l]).$$

1.2. **Related Work.** In [JLLZ21b] and [JLLZ21a], the authors prove the Kuznetsov component $\mathcal{K}u(X)$ of smooth Gushel-Mukai threefold X determine its birational isomorphism class. Moreover, the fiber of (categorical) period map for (Kuznetsov component) intermediate Jacobian of index one prime Fano threefold of genus $g \geq 6$ is computed in [JLLZ21b, Theorem 1.4]. (Birational) categorical Torelli theorems have been studied for many varieties. Interested reader could refer to [PS22] for more details.

1.3. **Organization of the article.** In Section 2 we recall terminology and basic property of semi-orthogonal decomposition(SOD) of bounded derived category of coherent sheaves. Then we introduce subcategory of homological finite objects and recall its compatibility with respect to SOD. In Section 3, we introduce several additive invariants for DG categories. In particular, we prove several propositions of topological K-theories for admissible subcategories $\mathcal{A} \subset D^b(X)$ with a singular projective variety X. In Section 4, we introduce a family of nodal Fano threefolds and describe their SOD following [KS22]. We prove birational Torelli theorems for one-nodal maximally non-factorial prime Fano threefolds. In Section 5, we generalize intermediate Jacobian for admissible subcategory of a bounded derived category $D^b(X)$ for smooth projective variety X in [Per22] to smooth and proper DG category and even to arbitrary DG category. Then we prove Theorem 1.4. In Section 6, we prove Theorem 1.5. In Section 7, we prove Theorem 1.8. Then we prove Proposition 7.9 on the existence of a family of gluing objects in a family of smooth and proper categories. Then we prove Theorem 1.9. In Section 8, we prove Theorem 1.10.

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2. Semi-orthogonal decomposition and Homological finite objects

We review some useful facts and results about semi-orthogonal decompositions, the subcategory of homological finite objects. Then we review the definition of Hochschild (co)homology and (periodical) cyclic homology. Background on triangulated categories and derived categories of coherent sheaves can be found in [Huy06]. Let X be a projective variety. From now on, for any $E, F \in D^b(X)$, define

$$\mathrm{RHom}^\bullet(E, F) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^i(E, F)[-i].$$

2.1. Exceptional collections and semi-orthogonal decompositions.

Definition 2.1. Let $\mathcal{D}$ be a k-linear triangulated category and $E \in \mathcal{D}$. We say that E is an *exceptional object* if $\mathrm{RHom}^\bullet(E, E) = k$. Now let $(E_1, \dots, E_m)$ be a collection of exceptional objects in $\mathcal{D}$. We say it is an *exceptional collection* if $\mathrm{RHom}^\bullet(E_i, E_j) = 0$ for $i > j$.

Definition 2.2. Let $\mathcal{T}$ be a triangulated category and $\mathcal{D}$ a full triangulated subcategory. We define the *right orthogonal complement* of $\mathcal{D}$ in $\mathcal{T}$ as the full triangulated subcategory

$$\mathcal{D}^\perp = \{E \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(F, E) = 0 \text{ for all } F \in \mathcal{D}\}.$$

The *left orthogonal complement* is defined similarly, as

$$^\perp\mathcal{D} = \{E \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(E, F) = 0 \text{ for all } F \in \mathcal{D}\}.$$

Definition 2.3. Let $\mathcal{T}$ be a triangulated category. We say a full triangulated subcategory $\mathcal{D} \subset \mathcal{T}$ is *admissible*, if the inclusion functor $i : \mathcal{D} \hookrightarrow \mathcal{T}$ has left adjoint i^* and right adjoint $i^!$.

Definition 2.4. Let $\mathcal{T}$ be a triangulated category, and $(\mathcal{D}_1, \dots, \mathcal{D}_m)$ be a collection of full admissible subcategories of $\mathcal{T}$. We say that $\mathcal{T} = \langle \mathcal{D}_1, \dots, \mathcal{D}_m \rangle$ is a *semiorthogonal decomposition* of $\mathcal{T}$ if $\mathcal{D}_j \subset \mathcal{D}_i^{\perp}$ for all $i > j$, and the subcategories $(\mathcal{D}_1, \dots, \mathcal{D}_m)$ generate $\mathcal{T}$, i.e. the category resulting from taking allcones of objects in the categories $(\mathcal{D}_1, \dots, \mathcal{D}_m)$ is equivalent to $\mathcal{T}$.

Let $\mathcal{T}$ admits the Serre functor $S_{\mathcal{T}}$. Then we have:

Proposition 2.5. *If $\mathcal{T} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ is a semi-orthogonal decomposition, then $\mathcal{T} \simeq \langle S_{\mathcal{T}}(\mathcal{D}_2), \mathcal{D}_1 \rangle \simeq \langle \mathcal{D}_2, S_{\mathcal{T}}^{-1}(\mathcal{D}_1) \rangle$ are also semi-orthogonal decompositions.*

Let $i\colon \mathcal{D} \hookrightarrow \mathcal{T}$ be an admissible triangulated subcategory. Then the *left mutation functor* $\mathbf{L}_{\mathcal{D}}$ through $\mathcal{D}$ is defined as the functor lying in the canonical functorial exact triangle

$$ii^! \to \mathrm{id} \to \mathbf{L}_{\mathcal{D}}$$

and the *right mutation functor* $\mathbf{R}_{\mathcal{D}}$ through $\mathcal{D}$ is defined similarly, by the triangle

$$\mathbf{R}_{\mathcal{D}} \to \mathrm{id} \to ii^*.$$

Therefore, $\mathbf{L}_{\mathcal{D}}$ is exactly the left adjoint functor of $\mathcal{D}^{\perp} \hookrightarrow \mathcal{T}$. Similarly, $\mathbf{R}_{\mathcal{D}}$ is the right adjoint functor of ${}^{\perp}\mathcal{D} \hookrightarrow \mathcal{T}$.

When $E \in \mathrm{D}^b(X)$ is an exceptional object, and $F \in \mathrm{D}^b(X)$ is any object, the left mutation $\mathbf{L}_E F$ fits into the triangle

$$E \otimes \mathrm{RHom}_X(E, F) \to F \to \mathbf{L}_E F,$$

and the right mutation $\mathbf{R}_E F$ fits into the triangle

$$\mathbf{R}_E F \to F \to E \otimes \mathrm{RHom}_X(F, E)^{\vee}.$$

A calculation of adjoint functors gives the following.

Lemma 2.6. *Let $\mathcal{T} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ be a semi-orthogonal decomposition. Then*

$$S_{\mathcal{D}_2} = \mathbf{R}_{\mathcal{D}_1} \circ S_{\mathcal{T}} \quad \textit{and} \quad S_{\mathcal{D}_1}^{-1} = \mathbf{L}_{\mathcal{D}_2} \circ S_{\mathcal{T}}^{-1}.$$

2.2. **Mutation.** Let $\pi : Y \longrightarrow X$ be a morphism with $\mathbb{R}\pi_* \mathcal{O}_Y = \mathcal{O}_X$. Assume Y is smooth, and there is a semi-orthogonal decomposition

$$\mathrm{D}^b(Y) = \langle \mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_m \rangle,$$

which induces a strong semi-orthogonal decomposition,

$$\mathrm{D}^b(X) = \langle \mathbb{R}\pi_* \mathcal{A}_1, \mathbb{R}\pi_* \mathcal{A}_2, \cdots, \mathbb{R}\pi_* \mathcal{A}_m \rangle.$$

We prove the following,

Theorem 2.7. *Let $E \in \mathrm{Perf}(X)$ that lies in some semi-orthogonal component and M_Y and M_X be the same type of mutations for the semi-orthogonal decompositions of $D^b(Y)$ and $D^b(X)$ respectively. We have*

$$\mathbb{R}\pi_* M_Y(\mathbb{L}\pi^* E) = M_X(E).$$

Proof. It suffices to prove the same statement for projection functor of semi-orthogonal decomposition since the mutation functor applying to E is certain projection functor with respect to a semi-orthogonal decomposition. Without loss of generality, we assume $m = 2$. Namely, it suffices to prove the following lemma.

Lemma 2.8. *Assume a semi-orthogonal decomposition*

$$\mathrm{D}^b(Y) = \langle \mathcal{A}_1, \mathcal{A}_2 \rangle$$

which induces a strong semi-orthogonal decomposition,

$$\mathrm{D}^b(X) = \langle \mathbb{R}\pi_*\mathcal{A}_1, \mathbb{R}\pi_*\mathcal{A}_2 \rangle.$$

Let $E \in \mathrm{Perf}(X)$. Let $P_{Y,i}$ and $P_{X,i}$ be the projection functor to the semi-orthogonal components, $i = 1, 2$. We have

$$\mathbb{R}\pi_* P_{Y,i}(\mathbb{L}\pi^* E) = P_{X,i}(E).$$

We prove the lemma for $i = 1$. Let

$$P_2 \to \Delta_Y \to P_1$$

be the triangle with respect to the semi-orthogonal decomposition of $D^b(Y)$. Clearly $\mathbb{L}\pi^*E \in D^b(Y)$. Then $P_{Y,1}(\mathbb{L}\pi^*E) = \Phi_{P_1}(\mathbb{L}\pi^*E)$. By projection formula, we have

$$\mathbb{R}\pi_*(\Phi_{P_1}(\mathbb{L}\pi^*E)) = \Phi_{\mathbb{R}(\pi,\pi)_*P_1}(E).$$

Since the triangle

$$\mathbb{R}(\pi,\pi)_*P_2 \to \mathbb{R}(\pi,\pi)_*\Delta_Y = \Delta_X \to \mathbb{R}(\pi,\pi)_*P_1$$

is the triangle for the semi-orthogonal decomposition of $Perf(X)$, we have $P_{X,1}(E) = \Phi_{\mathbb{R}(\pi,\pi)_*P_1}(E)$. Thus we have the statement of the lemma

$$\mathbb{R}\pi_* P_{Y,i}(\mathbb{L}\pi^* E) = P_{X,i}(E).$$

□

2.3. **Homologically finite subcategory.** For this subsection, we refer to [Orl].

Definition 2.9. We say that an object E in triangulated category $\mathcal{T}$ is homological finite if for any other object $F \in \mathcal{T}$ all $\mathrm{Hom}(E, F[i])$ are trivial except for finite number of $i \in \mathbb{Z}$. All such objects form a full subcategory of $\mathcal{T}$, denoted by $\mathcal{T}_{hf}$.

Proposition 2.10. *[Orl, Proposition 1.11] Let X be a quasi-projective variety. Then $D^b(X)_{hf} = \mathrm{Perf}(X)$.*

The next proposition shows that the subcategory of homologically finite objects is compatible with semi-orthogonal decomposition.

Proposition 2.11. *[Orl, Proposition 1.10] Let $\mathcal{T} = \langle \mathcal{N}_1, \mathcal{N}_2, \cdots \mathcal{N}_j \rangle$ be a strong semi-orthogonal decomposition, namely the embedding of $\mathcal{N}_i \hookrightarrow \mathcal{T}$ is admissible. Then there is a semi-orthogonal decomposition,*

$$\mathcal{T}_{hf} = \langle \mathcal{N}_{1_{hf}}, \mathcal{N}_{2_{hf}}, \cdots \mathcal{N}_{j_{hf}} \rangle.$$

2.4. **Del Pezzo threefolds of Picard rank one and Kuznetsov components.** In this section, we briefly review Kuznetsov components of del Pezzo threefold of Picard rank one. By [Isk80] every del Pezzo threefold $Y := Y_d$ of rank one and degree d belongs to the following five families, indexed by their degree $d = H^3 = \{1, 2, 3, 4, 5\}$:

(1) $Y_5 = \mathbb{P}^6 \cap \mathrm{Gr}(2, 5)$ is a codimension 3 linear section of Grassmannian $\mathrm{Gr}(2, 5)$.
(2) $Y_4 = Q \cap Q'$ is intersection of two quadric hypersurfaces in $\mathbb{P}^5$.
(3) $Y_3 \subset \mathbb{P}^4$ is cubic threefold.
(4) Y_2 is a quartic double solid, i.e. a double cover of $\mathbb{P}^3$ with smooth branch divisor $R \in |\mathcal{O}_{\mathbb{P}^3}(4)|$.
(5) Y_1 is a degree 6 hypersurface of weighted projective space $\mathbb{P}(1, 1, 1, 2, 3)$.

Every del Pezzo threefold Y above admits the semi-orthogonal decomposition

$$D^b(Y) = \langle \mathcal{K}u(Y), \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle,$$

where the Kuznetsov component $\mathcal{K}u(Y)$ is defined to be the semi-orthogonal component which is the right orthogonal complement of an exceptional collection $\mathcal{O}_Y, \mathcal{O}_Y(1)$. We can identify the numerical Grothendieck group $\mathcal{N}(\mathcal{K}u(Y))$ of $\mathcal{K}u(Y)$ with the image of Chern character map

$$\mathrm{ch}\colon \mathcal{N}(\mathcal{K}u(Y)) \to H^*(X, \mathbb{Q}).$$

It is a rank 2 lattice spanned by the classes

$$\mathbf{v} = \left(1,\ 0,\ -\frac{1}{d}H^2,\ 0\right) \qquad \text{and} \qquad \mathbf{w} = \left(0,\ H,\ -\frac{1}{2}H^2,\ \left(\frac{1}{6} - \frac{1}{d}\right) H^3\right).$$

With respect to this basis, the Euler form on $\mathcal{N}(\mathcal{K}u(Y))$ is represented by the matrix

$$\begin{pmatrix} -1 & -1 \\ 1-d & -d \end{pmatrix}. \tag{1}$$

3. Additive invariants for DG categories

3.1. **Hochschild homology, negative cyclic homology and periodical cyclic homology of DG categories.** In this subsection, we give a brief introduction to Hochschild homology, cyclic homology, and the Hodge theory of smooth proper *dg* categories. The reference is [Kel07].

Definition 3.1. A $\mathbb{Z}$-graded *DG* category is a category whose morphism space $\mathrm{Hom}(E, F)$ are complexes of k module, and the composition

$$\mathrm{Hom}(E_1, E_2) \otimes \mathrm{Hom}(E_2, E_3) \to \mathrm{Hom}(E_1, E_3).$$

is morphism of chain complexes. The unit morphism is of closed degree 0.

Example 3.2. A basic example is the category $C_{dg}(k)$ whose objects are complexes of k module and the morphism spaces are the internal Hom complexes.

A derived category $D(\mathcal{A})$ is the localization of the category of right $\mathcal{A}$ over the quasi-isomorphisms. A *dg* functor $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is a Morita-equivalence if the induced funtor on derived category $\phi_* : D(\mathcal{A}_1) \to D(\mathcal{A}_2)$ is an exact equivalence. Let dg-cat be the category of *dg* categories whose morphism spaces are *dg* functors. There is a model structure with weak equivalence to be the Morita equivalences. We write $\mathrm{Hmo}(\mathrm{dg}-\mathrm{cat})$ the homotopical category with respect to the Morita equivalences.

Definition 3.3. An additive invariant of *dg* categories is an additive functor of additive categories $A : \mathrm{dg}-\mathrm{cat} \to \mathcal{C}$ such that

- the functor A factors through Hmo(dg-cat), namely, A maps Morita equivalence to isomorphism.
- the functor A is additive with respect to semi-orthogonal decomposition.

The natural additive invariant of the *DG* category is Hochschild homology and cyclic homology.

Example 3.4. Let $C(\mathcal{A}) = \mathcal{A} \otimes^{\mathbb{L}}_{\mathcal{A}^{op} \otimes \mathcal{A}} \mathcal{A}$. Using bar resolution of bi-module $\mathcal{A}$, complex $C(\mathcal{A})$ is a mixed complex, namely, it is a $\mathbb{Z}$-graded vector space with differential b and the Connes B operator such that $bB + Bb = 0$. The derived category of the mixed complexes is equivalent to the derived category of dg algebra $D(\Lambda)$, where $\Lambda = \frac{k[\epsilon]}{\epsilon^2}$, degree of ϵ is -1 and $d(\epsilon) = 0$. We have mixed complexes as additive invariant

$$\mathrm{Mix} : \mathrm{Hmo}(\mathrm{dg}-\mathrm{cat}) \to D(\Lambda)$$

- (Hochschild homology) $\mathrm{HH}_*(\mathcal{A}) = \mathrm{H}^*(C(\mathcal{A}), b)$.
- (Periodic cyclic homology) $\mathrm{HP}_*(\mathcal{A}) = \mathrm{H}^*(C(\mathcal{A})((u)), b + uB)$. Here u is a formal variable of degree 2.
- (Negative cyclic homology) $\mathrm{HN}_*(\mathcal{A}) = \mathrm{RHom}_\Lambda(k, C(\mathcal{A}))$.

The negative cyclic homology and periodic cyclic homology are additive invariants.

A *dg* category is smooth and proper if it is proper and $\mathcal{A}$ is perfect over $\mathcal{A}^{op}\otimes\mathcal{A}$. The examples are natural *dg* enhancement of admissible subcategory of derived category of smooth proper varieties. There are similar Hodge filtration of $\mathrm{HP}_*(\mathcal{A})$ (which is de Rham cohomology when $\mathcal{A}$ is a derived category of smooth proper variety)

$$\subset \cdots \mathrm{HN}_{-2j-1}(\mathcal{A}) \subset \cdots \subset \mathrm{HN}_{-1}(\mathcal{A}) \subset \mathrm{HN}_1(\mathcal{A}) \subset \mathrm{HN}_3(\mathcal{A}) \cdots \subset \mathrm{HN}_{2j+1}(\mathcal{A}) \subset \cdots \subset \mathrm{HP}_1(\mathcal{A}).$$

$$\subset \cdots \mathrm{HN}_{-2j}(\mathcal{A}) \subset \cdots \subset \mathrm{HN}_{-2}(\mathcal{A}) \subset \mathrm{HN}_0(\mathcal{A}) \subset \mathrm{HN}_2(\mathcal{A}) \cdots \subset \mathrm{HN}_{2j}(\mathcal{A}) \subset \cdots \subset \mathrm{HP}_0(\mathcal{A}).$$

Thanks to [Kal08] and [Kal17], the noncommutative Hodge to de Rham spectral sequence degenerates. Therefore we have exact sequences,

$$0 \to \mathrm{HN}_{2j-1}(\mathcal{A}) \to \mathrm{HN}_{2j+1}(\mathcal{A}) \to \mathrm{HH}_{2j+1}(\mathcal{A}) \to 0.$$

$$0 \to \mathrm{HN}_{2j}(\mathcal{A}) \to \mathrm{HN}_{2j+2}(\mathcal{A}) \to \mathrm{HH}_{2j+2}(\mathcal{A}) \to 0.$$

If $\mathcal{A}$ is an admissible subcategory of derived category of smooth projective variety, then the exact sequences have natural splitting induced from Hodge decomposition.

3.2. **Topological K-theory of admissible subcategories.**

Definition 3.5. [Bla16] The topological K-theory for dg categories over $\mathbb{C}$ is an additive invariant,

$$\mathrm{K}_1^{top} : \mathrm{Hmo}(\mathrm{dg} - cat) \to \mathbb{Z} - mod.$$

with Chern character map

$$\mathrm{ch}^{top} : \mathrm{K}_1^{top}(\mathcal{A}) \to \mathrm{HP}_1(\mathcal{A}).$$

Furthermore $\mathrm{K}_1^{top}(D_{dg}^{perf}(X)) \otimes \mathbb{C} \cong \mathrm{H}^{\mathrm{odd}}(X, \mathbb{C})$, and the Chern character is the usual Chern character.

Let X be a singular projective variety and $\mathcal{A} \subset D^b(X)$ be the admissible subcategory and we have the semi-orthogonal decomposition $D^b(X) = \langle \mathcal{A}, {}^\perp\mathcal{A}\rangle$. Let $\mathcal{A}_{\mathrm{hf}} \subset \mathcal{A}$ the subcategory of homological finite objects of $\mathcal{A}$, then by Proposition 2.11, we have the semi-orthogonal decomposition $D^{perf}(X) = \langle \mathcal{A}_{\mathrm{hf}}, {}^\perp\mathcal{A}_{\mathrm{hf}}\rangle$. We aim at extracting Hodge theory from these $\mathcal{A}_{\mathrm{hf}}$, with examples considered in this article. Since $\mathcal{A}_{\mathrm{hf}}$ is proper we have natural Euler paring for $\mathrm{K}_1^{top}(\mathcal{A}_{\mathrm{hf}})$ [Per22, Lemma 5.2].

Lemma 3.6. *Let $\mathcal{A} \subset D^b(X)$ be an admissible subcategory with X a projective variety. If $\mathrm{HP}_1(\mathcal{A}_{\mathrm{hf}}) = 0$, then $\mathrm{K}_1^{top}(\mathcal{A}_{\mathrm{hf}})$ is a torsion abelian group.*

Proof. Since we have injective map $\mathrm{K}_1^{top}(D_{dg}^{perf}(X))\otimes\mathbb{C} \xhookrightarrow{Ch^{top}} \mathrm{HP}_1(D_{dg}^{perf}(X))$ [Bla16, Proposition 4.32], by additivity of K_1^{top} and HP_1 with respect to semi-orthogonal decomposition, we have injective map

$$\mathrm{K}_1^{top}(\mathcal{A}_{\mathrm{hf}}) \otimes \mathbb{C} \hookrightarrow \mathrm{HP}_1(\mathcal{A}_{\mathrm{hf}}) = 0.$$

Therefore $K_1^{top}(\mathcal{A}) \otimes \mathbb{C} = 0$, and then $K_1^{top}(\mathcal{A}_{\mathrm{hf}})$ is a torsion abelian group. ☐

Lemma 3.7. *Let $\mathcal{A} \subset D^b(X)$ be an admissible subcategory, where X is a projective variety. Let $\mathcal{A} = \langle \mathcal{A}_1, \mathcal{B}\rangle$ be an admissible semi-orthogonal decomposition, and $\mathrm{HP}_1(\mathcal{B}_{\mathrm{hf}}) = 0$, then we have an isometry $\mathrm{K}_1^{top}(\mathcal{A}_{\mathrm{hf}})_{tf} \cong \mathrm{K}_1^{top}(\mathcal{A}_{1,\mathrm{hf}})_{tf}$ with respect to Euler pairing.*

Proof. We have an isomorphism

$$\mathrm{K}_1^{top}(\mathcal{A}_{\mathrm{hf}}) \cong \mathrm{K}_1^{top}(\mathcal{A}_{1,\mathrm{hf}}) \oplus K_1^{top}(\mathcal{B}_{\mathrm{hf}}).$$

First, if v' is torsion in $\mathrm{K}_1^{top}(\mathcal{A}_{\mathrm{hf}})$, then there exist integer m, $mv' = 0$, and $m\mathcal{X}(v, v') = \mathcal{X}(v, mv') = 0$, hence $\mathcal{X}(v, v') = 0$. That is to say, the torsion elements have no contribution to the Euler pairing. Thus, we have an isometry $\mathrm{K}_1^{top}(\mathcal{A}_{\mathrm{hf}})_{tf} \cong \mathrm{K}_1^{top}(\mathcal{A}_{1,\mathrm{hf}})_{tf}$ by Lemma 3.6. □

4. One nodal maximally non-factorial Fano threefold and derived category

In this section we review definition of one nodal maximally non-factorial Fano threefolds and its property following [CKGS23]. Then we briefly review the work [KS22] and [KS23] on derived category and semi-orthogonal decomposition of such nodal Fano threefolds. Then we prove (birational) Torelli theorem for these Fano threefolds.

Let X be a Fano threefold that has at worst isolated ordinary double points(nodes). Let $\pi : \widetilde{X} \to X$ be the blow up at singular locus with exceptional divisors $E_1, \dots, E_r$. For each i, we have $E_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{O}_{E_i}(E_i) = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$. Thus $\mathrm{Pic}(E_i)/[\mathcal{O}_{E_i}(E_i)] \cong \mathbb{Z}$.

Definition 4.1. A nodal threefold X is called maximally non-factorial if the morphism

$$\mathrm{Pic}(\widetilde{X}) \to \bigoplus_{i=1}^{r} \mathrm{Pic}(E_i) \to \bigoplus_{i=1}^{r} (\mathrm{Pic}(E_i)/[\mathcal{O}_{E_i}(E_i)]) \cong \mathbb{Z}^r$$

is surjective.

The next proposition leads to the definition of intermediate Jacobian of a one nodal maximally non-factorial Fano threefold.

Proposition 4.2. *[KS23, Proposition A.16] Let $f : \mathcal{X} \to B$ be a smoothing of a one nodal maximally non-factorial Fano threefold X. Then there is a smooth and proper family $\mathcal{J} \to B$ of principally polarized abelian varieties such that*

$$\mathcal{J}_b \cong \begin{cases} \mathrm{Jac}(\mathcal{X}_b), & b \neq o \\ \mathrm{Jac}(\hat{X}), & b = o. \end{cases}$$

where $\hat{X}$ is a small resolution of X and these isomorphisms are compatible with principal polarizations.

Definition 4.3. Let X be an one nodal maximally non-factorial prime Fano threefold. Its intermediate Jacobian $J(X) := \mathcal{J}_o$, where $\mathcal{J}$ is the smooth and proper family of intermediate Jacobian obtained in Proposition 4.2.

Definition 4.4. An object $P \in D^b(X)$ is $\mathbb{P}^{\infty,q}$-object if $\mathrm{Ext}^\bullet(P, P) \cong k[t]$, where $\deg(t) = q$.

In [KP23] and [KS22] the authors introduce the notion of *Categorical absorption* of singularties. Briefly to say, for a nodal complex algebraic variety X, they managed to establish the semi-orthogonal decomposition of the bounded derived category $D^b(X)$ as

$$D^b(X) = \langle \mathcal{P}_1, \dots, \mathcal{P}_r, \mathcal{A} \rangle,$$

where $\mathcal{P}_i$ is generated by $\mathbb{P}^\infty$ objects for each i, which are responsible for nodal singularities and their orthogonal complement $\mathcal{A}$ is a smooth and proper category. In the case of one nodal maximally non-factorial Fano threefold of Picard rank one, they prove the following theorem.

Theorem 4.5. *[Xie23][KS23][KP23] Let X be a one nodal maximally non-factorial prime Fano threefold, then*

(1) If X is a nodal quintic del Pezzo threefold, then

$$D^b(X) = \langle D^b(R_{m-1}), D^b(R_1), \mathcal{O}_X, \mathcal{O}_X(1)\rangle,$$

where $1 \leq m \leq 3$ is the number of nodes of X and R_m is a quiver algebra defined in [Xie23, (1.2)]. Moreover, $D^b(R_n)$ admits a further semi-orthogonal decomposition with a copy of $D^b(k)$ and n-copies of $D^b(A)$ with $A = \frac{\mathbb{K}[x]}{(x^2)}$ is a DG-algebra with $\deg(x) = -1$.

(2) If X is an one nodal maximally non-factorial prime Fano threefold of genus $g \geq 6$ and $g \neq 9, 10$, then

$$D^b(X) = \langle \mathcal{K}u(X), \mathcal{O}_X, \mathcal{U}_X^\vee\rangle,$$

such that

$$\mathcal{K}u(X) = \begin{cases} \langle \mathcal{K}u(Y_2), \mathcal{P}\rangle, & g = 6; \\ \langle D^b(C_7), \mathcal{P}\rangle, & g = 7; \\ \langle \mathcal{K}u(Y_3), \mathcal{P}\rangle, & g = 8; \\ \langle \mathcal{B}, \mathcal{P}\rangle, & g = 12, \end{cases}$$

where C_7 is a smooth curve of genus $g = 7$ respectively, and $\mathcal{B} = \langle E_1, \ldots, E_4\rangle$ is a category generated by an exceptional collection of vector bundles. Moreover $\mathcal{U}_X^\vee$ is the Mukai bundle for each genus and the category $\mathcal{P}$ is derived equivalent to $D^b(A)$.

Next we prove (birational) Torelli theorem for nodal Fano threefolds as above.

Theorem 4.6. *Let X and X' be 1-nodal maximally non-factorial index one prime Fano threefold of genus $g \geq 6$ such that $J(X) \cong J(X')$ as principal polarised abelian varieties, then $X \simeq X'$.*

Proof. The classification of one nodal maximally non-factorial index one prime Fano threefolds are given in [CKGS23]. By [CKGS23, Remark], those 1-nodal Fano threefolds of genus $g = 7, 9, 10, 12$ are all rational, therefore the statement is trivial. We start with $g = 6$, by [CKGS23], it is given by the bridge construction

$$\begin{array}{ccc} & \widetilde{Y} & \\ {\scriptstyle\sigma}\swarrow & & \searrow{\scriptstyle\pi} \\ Y & & X \end{array}$$

with quartic double solid Y and $\widetilde{Y}$ the blow up of Y along a line. Note that π is a small contraction along a line. Then by [KS23, Proposition A.16], $J(X) \cong J(\widetilde{Y}) \cong J(Y)$ and $J(X') \cong J(Y')$, thus $J(Y) \cong J(Y')$ as polarised abelian varieties, then by [Deb12, Section 3.5], $Y \cong Y'$, thus $X \simeq X'$ since σ and π are birational maps.

Next, we consider the $g = 8$ case. By [CKGS23], there are two types of $g = 8$ prime Fano threefolds, both of which are given by bridge construction

$$\begin{array}{ccc} & \widetilde{Y} & \\ {\scriptstyle\sigma_1}\swarrow & & \searrow{\scriptstyle\pi_1} \\ Y & & X \end{array}$$

$$\begin{array}{ccc} & \widetilde{Y} & \\ {\scriptstyle\sigma_2}\swarrow & & \searrow{\scriptstyle\pi_2} \\ \mathbb{P}^2 & & X \end{array}$$

where σ_1 is blow up of a cubic threefold Y along a conic and σ_2 is a conic bundle with quintic discriminant curve. For the first case, similar to the Gushel-Mukai case $J(X) \cong J(\widetilde{Y}) \cong J(Y)$. For the second case, by the commutative diagram in [CKGS23, **Construction**] and chain of equalities on $h^{2,1}$, we have $J(\widetilde{Y'}) \cong J(Y')$. Thus, if $J(X) \cong J(X')$ as principal polarised abelian varieties, then $J(Y) \cong J(Y')$ as principal polarised abelian varieties, where Y, Y' are cubic threefolds, then $Y \cong Y'$ by Torelli theorem for cubic threefolds(cf. [Deb12, Section 3.4]). For the first case, $X \simeq Y$. For the second case, X' is also birational to Y'(from the same commutative diagram above). Therefore $X \simeq X'$. □

In the case of smooth prime Fano threefold of index one and genus 7, the actual Torelli theorem holds. One might wonder if the statement extends to the nodal case. We provide a partial answer to this question. Recall any genus 7 1-nodal maximally non-factorial prime Fano threefold is constructed via the *bridge construction* from $\mathbb{P}^3$:

$$\begin{array}{ccc} & \widetilde{\mathbb{P}^3} & \\ {}^{\sigma}\swarrow & & \searrow^{\pi} \\ \mathbb{P}^3 & & X \end{array}$$

where σ is blow up a genus 7 and degree 8 curve in $\mathbb{P}^3$ and π is a small contraction induced by anticanonical linear system $|-K_{\widetilde{\mathbb{P}^3}}|$. Recall the locus of non-hyperelliptic curves which have a map of degree four to $\mathbb{P}^1$ is called tetragonal loci. If C is general, then by [Ma13], there is a unique g^1_4 on C.

Proposition 4.7. *Let X and X' be 1-nodal maximally non-factorial prime Fano threefold of genus 7, assume the genus 7 and degree 8 curve $C \subset \mathbb{P}^3$ is general in tetragonal loci. Then $J(X) \cong J(X') \Longrightarrow X \cong X'$.*

Proof. Now $J(X) \cong J(X')$ implies $J(C) \cong J(C')$, which implies $C \cong C'$. Now we show that if $C, C' \in \mathcal{T}_7$(tetragonal loci) are *general*, then the isomorphism of C and C' is induced from that on $\mathbb{P}^3$. Indeed, since $C \subset \mathbb{P}^3$, there is a line bundle $\mathcal{L}_1$ of degree 8 and $h^0(\mathcal{L}_1) = 4$. Let $\mathcal{L}'_1 := \omega_C \otimes \mathcal{L}_1^{-1}$, then degree of $\mathcal{L}'_1$ is $12 - 8 = 4$ and $h^0(\mathcal{L}'_1) = h^1(\mathcal{L}_1)$ By Serre duality. Then by Riemann-Roch theorem we have $h^0(\mathcal{L}_1) - h^1(\mathcal{L}_1) = \deg(\mathcal{L}_1) + 1 - g = 2$, then $h^0(\mathcal{L}'_1) = 2$, thus $\mathcal{L}'_1$ is a g^1_4. But $C \in \mathcal{T}_g$ is general, then g^1_4 is unique by [Ma13, Section 3], therefore $\mathcal{L}_1$ is unique. This means that C determines $\mathcal{L}_1$. Thus if $C' \cong C$, then $\mathcal{L}_2 \cong \mathcal{L}_1$ and both line bundles are the ones which determine their embedding to $\mathbb{P}^3$, then the isomorphism of C and C' can be lifted to the one in PGL_4, thus $X \cong X'$. □

5. Intermediate Jacobian of DG category: Hodge theory of Kuznetsov components

5.1. **Intermediate Jacobian of DG categories.** In this section, we generalize the construction of intermediate Jacobian for admissible subcategory of bounded derived category of a smooth projective variety in [Per22, Definition 5.24] to arbitrary DG category and prove Theorem 1.4.

Let $\mathcal{A} \subset D^b(Z)$ be an admissible subcategory, where Z is smooth projective. First we review the construction of $\mathrm{J}(\mathcal{A})$ in [Per22, Definition 5.24]. There is a Hodge filtration

$$\cdots \subset \mathrm{HN}_{-3}(\mathcal{A}) \subset \mathrm{HN}_{-1}(\mathcal{A}) \subset \mathrm{HN}_1(\mathcal{A}) \subset \cdots \subset \mathrm{HP}_1(\mathcal{A}).$$

The natural splitting from that of Z gives a weight one Hodge structure for topological K-group $\mathrm{K}_1^{top}(\mathcal{A})_{tf}$. Namely, the topological Chern character induces,

$$\mathrm{K}_1^{top}(\mathcal{A})_{tf}\otimes\mathbb{C}\cong^{\mathrm{ch}^{top}}\mathrm{HP}_1(\mathcal{A})\cong\mathrm{HN}_{-1}(\mathcal{A})\oplus\overline{\mathrm{HN}_{-1}(\mathcal{A})}.$$

Thus, we have a complex torus associated to this weight one Hodge structure. More explicitly,

$$\mathrm{J}(\mathcal{A})=\frac{\mathrm{HP}_1(\mathcal{A})}{\mathrm{HN}_{-1}(\mathcal{A})+\mathrm{Imch}\,\mathrm{K}_1^{top}(\mathcal{A})}.$$

Let X be a smooth Fano threefold such that the bounded derived category $D^b(X)$ admits a semi-orthogonal decomposition

$$D^b(X)=\langle\mathcal{K}u(X),E_1,\dots,E_n\rangle,$$

where $\langle E_1,E_2,\cdots,E_n\rangle$ is an exceptional collection, then by [JLLZ23, Lemma 3.9], the intermediate Jacobian $\mathrm{J}(\mathcal{K}u(X))\cong J(X)$.

Next we generalize the definition of intermediate Jacobian for admissible subcategory $\mathcal{A}$ to arbitrary dg category.

Definition 5.1. Let $\mathcal{B}$ be a dg category. We have a natural map $j:\mathrm{HN}_{-1}(\mathcal{B})\to\mathrm{HP}_1(\mathcal{B})$. Define intermediate Jacobian

$$\mathrm{J}(\mathcal{B})=\frac{\mathrm{H}P_1(\mathcal{B})}{j(\mathrm{HN}_{-1}(\mathcal{B}))+\mathrm{ImCh}\,\mathrm{K}_1^{top}(\mathcal{B})}.$$

In this case, intermediate Jacobian $\mathrm{J}(\mathcal{B})$ is only an abelian group. Next lemma shows that the abstract intermediate Jacobian construction is additive with respect to semi-orthogonal decomposition.

Lemma 5.2. *Let $\mathcal{A}$, $\mathcal{A}_1$, and $\mathcal{A}_2$ be dg categories. Let $\mathcal{A}=\langle\mathcal{A}_1,\mathcal{A}_2\rangle$ be a semi-orthogonal decomposition. Then $J(A)=J(\mathcal{A}_1)\oplus J(\mathcal{A}_2)$. In particular, if $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}$ are admissible subcategory of some derived categories of smooth projective varieties, then it is a sum of complex torus.*

Proof. Since periodic cyclic homology, negative cyclic homology, and topological K-theory are additive invariants, $J(\mathcal{A})\cong J(\mathcal{A}_1)\oplus J(\mathcal{A}_2)$. □

Lemma 5.3. *Let $A=\frac{k[x]}{(x^2)}$ be a dg algebra over k with* $\mathrm{degree}(x)=-1$. *Then* $\mathrm{HP}_1(A)=0$ *and* $\mathrm{K}_1^{top}(A)$ *is a torsion abelian group.*

Proof. By [Lod13, Corollary 5.3.14], $\mathrm{HP}(A)\cong\mathrm{HP}(k)$, since k is a field of characteristic 0. Then by [Lod13, Section 5.1.4], $\mathrm{HP}_{2n-1}=0$ for any n, thus in particular $\mathrm{HP}_1(A)=\mathrm{HP}_1(k)=0$. According to Theorem 4.5, $D^{perf}(A)$ is an admissible subcategory of some projective variety. According to Lemma 3.6, if $\mathrm{HP}_1(A)=0$, then $\mathrm{K}_1^{top}(A)$ is a torsion abelian group. □

5.2. **Intermediate Jacobian of nodal prime Fano threefolds.** Let X be an 1-nodal maximally non-factorial index one prime Fano threefold of genus $g\geq 6$, then there is a semi-orthogonal decomposition,

$$D^b(X)=\langle\mathcal{K}u(X),\mathcal{O}_X,\mathcal{U}_X^{\vee}\rangle.$$

Then by [KS23] and [KP23], the Kuznetsov component $\mathcal{K}u(X)$ has further semi-orthogonal decomposition $\mathcal{K}u(X)=\langle\mathcal{P},\mathcal{A}\rangle$, where $\mathcal{A}\simeq\mathcal{K}u(Y)$, for a smooth del Pezzo threefold Y if genus $g=6,8,12$. And if $g=7$, then $\mathcal{K}u(Y)\simeq D^b(C_7)$ for a smooth curve C_7 of genus 7.

Theorem 5.4. *Let X be a one-nodal maximally non-factorial prime Fano threefold of index one and genus $g = 6, 7, 8, 12$. Then*

$$\mathrm{J}(\mathcal{K}u(X)_{\mathrm{hf}}) \cong J(X).$$

In particular, if X is an index one nodal prime Fano threefold of degree $4d+2$ such that $d = 2, 3, 5$, then

$$\mathrm{J}(\mathcal{K}u(X_{4d+2})_{\mathrm{hf}}) \cong J(Y_d).$$

Proof. Let X be a nodal prime Fano threefold as above, then $\mathcal{K}u(X) = \langle \mathcal{P}, \mathcal{A} \rangle$, where $\mathcal{P}$ is generated by a $\mathbb{P}^{\infty,2}$-object P such that $\langle P \rangle \simeq D^b(A)$, where $A = \frac{k[x]}{(x^2)}$ is a dg algebra over k with degree$(x) = -1$. Then by Proposition 2.11,

$$\mathcal{K}u(X)_{\mathrm{hf}} = \langle \mathcal{A}_{\mathrm{hf}}, \langle P \rangle_{\mathrm{hf}} \rangle.$$

By [KS23, Proposition 3.3], $\langle P \rangle_{\mathrm{hf}} \simeq \mathrm{Perf}(A)$. Then apply intermediate Jacobian J in Definition 5.1 and by Lemma 5.2, we get

$$\mathrm{J}(\mathcal{K}u(X)_{\mathrm{hf}}) \cong \mathrm{J}(\mathcal{A}_{\mathrm{hf}}) \oplus \mathrm{J}(\mathrm{Perf}(A)),$$

where $\mathrm{J}(\mathrm{Perf}(A)) \cong 0$ by Lemma 5.3. By Lemma 3.7, we have an isomorphism of principle polarized abelian varieties

$$\mathrm{J}(\mathcal{K}u(X)_{\mathrm{hf}}) \cong \mathrm{J}(\mathcal{A}_{\mathrm{hf}}).$$

- If X is a 1-nodal maximally non-factorial prime Fano threefold of index one and genus $g = 6, 8$, then $\mathcal{A}_{\mathrm{hf}} = \mathcal{K}u(Y_d)_{\mathrm{hf}} = \mathcal{K}u(Y_d)$, where Y_d is a smooth del Pezzo threefold of degree $d = \frac{g-2}{2}$. Then $\mathrm{J}(\mathcal{A}_{\mathrm{hf}}) \cong \mathrm{J}(\mathcal{K}u(Y_d)) \cong J(Y_d)$ by [JLLZ23, Lemma 3.9].
- If X is a 1-nodal maximally non-factorial prime Fano threefold of index one genus $g = 12$, then by Theorem 4.5, $\mathcal{A}_{\mathrm{hf}} = \mathcal{B}$, where $\mathcal{B}$ is generated by an exceptional collection of vector bundles of length 4. Then $\mathrm{J}(\mathcal{B}) \cong 0 \cong J(Y_5)$.
- If X is a 1-nodal maximally non-factorial prime Fano threefold of index one and genus $g = 7$ respectively. Then by Theorem 4.5, $\mathcal{A}_{\mathrm{hf}} = D^b(C)$ for some smooth curve of genus 7. Then $\mathrm{J}(\mathcal{A}_{\mathrm{hf}}) \cong J(C)$.

For all cases above, there is a *Bridge construction* for X, which is listed at the end of the article [CKGS23]. Inspecting each case and looking at the proof of Theorem 4.6, we have the isomorphism $J(\mathcal{A}_{\mathrm{hf}}) \cong J(X)$. □

Remark 5.5. The reason that we exclude the cases $g = 9, 10$ is it is not clear that the absorption categories for these nodal Fano threefolds have the form as in Theorem 1.2.

5.3. **Intermediate Jacobians for nodal del Pezzo threefolds.** Let Y be a nodal del Pezzo threefold of degree 5. By [Xie23, Theorem 4.8], it admits the semi-orthogonal decomposition

$$D^b(Y) = \langle D^b(R_{m-1}), D^b(R_1), \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle,$$

where $1 \leq m \leq 3$ is the number of nodes of Y and R_m is a quiver algebra defined in [Xie23, (1.2)]. It is known that the bounded derived category $D^b(R_n)$ admits a further semi-orthogonal decomposition with a copy of $D^b(k)$ and n-copies of $D^b(\frac{\mathbb{K}[x]}{(x^2)})$, where $A := \frac{\mathbb{K}[x]}{(x^2)}$ is a DG-algebra with $\deg(x) = -1$. The obvious choice of the Kuznetsov component is the right orthogonal complement of $\mathcal{O}_Y, \mathcal{O}_Y(1)$. Then its subcategory of homological finite objects $\mathcal{K}u(Y)_{\mathrm{hf}}$ admits the semi-orthogonal decomposition with components as copies of $D^b(k)$ and $D^{perf}(A)$. By similar arguments in Theorem 5.4, we get $\mathrm{J}(\mathcal{K}u(Y)_{\mathrm{hf}}) = 0$. On the other hand, by [KS22, Proposition 6.19], the quintic nodal del Pezzo threefold Y admits a small resolution $\hat{Y}$ whose bounded derived category has a full exceptional collection, thus $J(\hat{Y}) = 0$. Then $J(Y) \cong J(\hat{Y}) = 0$. Therefore we have

$$\mathrm{J}(\mathcal{K}u(Y)_{\mathrm{hf}}) \cong J(Y).$$

5.4. **Intermediate Jacobian of a threefold with a non-isolated singularity.** We recall an example of projective threefold with non-isolated singularity in [Var24, Section 4]. Let $B = \mathbb{P}^1_{x:y} \times \mathbb{P}^1_{s:t} \times \mathbb{P}^1_{u:v}$ and a curve $\mathcal{C}$ as the vanishing locus $V(x, s^2u)$. Then $\mathcal{C}$ is a copy of $\mathbb{P}^1$ intersecting the thickened branch $V(s^2) \cong \mathbb{P}^1_{u:v} \times \frac{\mathbb{C}[s]}{s^2}$ at a fat point. Let X be the blow up of B at the curve $\mathcal{C}$. Thus X is singular with a line of surface nodes compounded with a threefold nodal singularity at one point and this singularity is non-isolated. Then there is a semi-orthogonal decomposition

$$D^b(X) = \langle D^b(\mathcal{C}), D^b(B)\rangle = \langle D^b(\mathbb{P}^1_{s:t}), \mathcal{D}, D^b(B)\rangle,$$

where $\mathcal{D}$ is the absorbing category, which is equivalent to $D^b(\frac{k[w,r]}{(w^2,r^2)}) \simeq D^b(\frac{k[w]}{(w^2)} \otimes \frac{k[r]}{(r^2)})$, where degree of w is 0 and degree r is -1. We define $\mathcal{K}u(X) := \langle D^b(\mathbb{P}^1_{s:t}), \mathcal{D}\rangle$. Then $\mathcal{K}u(X)_{\mathrm{hf}} \simeq \langle D^b(\mathbb{P}^1), D^{perf}(A)\rangle$, where A is $\frac{k[w,r]}{(w^2,r^2)}$ with degree w is 0 and degree r is -1. Since $\mathrm{HP}_1(A) \cong \mathrm{HP}_1(\frac{k[w]}{(w^2)}) \otimes \mathrm{HP}_1(\frac{k[r]}{(r^2)}) = 0$ by Lemma 5.3, we have $\mathrm{J}(\mathcal{K}u(X)_{\mathrm{hf}}) = 0$. But now it is not clear to us how to compute intermediate Jacobian of X.

6. Application: (Birational) Categorical Torelli Theorems

6.1. **(Birational) categorical Torelli theorem for nodal Fano threefolds.** In this section, we prove Theorem 1.5. First we show the intermediate Jacobians of the nodal Fano threefolds considered in our article are determined by their Kuznetsov components.

Proposition 6.1. *Let X and X' be 1-nodal maximally non-factorial prime Fano threefold of index one and genus $g = 6, 7, 8, 12$ such that $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$, then $J(X) \cong J(X')$ as principal polarised abelian varieties.*

Proof. Firstly, the equivalence $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$ induces an equivalence $\mathcal{K}u(X)_{\mathrm{hf}} \simeq \mathcal{K}u(X')_{\mathrm{hf}}$. Then, we have

$$\langle \mathcal{K}u(Y), \mathcal{P}\rangle_{\mathrm{hf}} \simeq \langle \mathcal{K}u(Y'), \mathcal{P}'\rangle_{\mathrm{hf}}.$$

According to Proposition 2.11, we have

$$\langle \mathcal{A}_{\mathrm{hf}}, \langle P\rangle_{\mathrm{hf}}\rangle \simeq \mathcal{A}'_{\mathrm{hf}}, \langle P'\rangle_{\mathrm{hf}}\rangle.$$

Then apply intermediate Jacobian in Definition 5.1, Lemma 5.2 and Lemma 5.3 we get

$$\mathrm{J}(\mathcal{A}_{\mathrm{hf}}) \oplus \mathrm{J}(A) \cong \mathrm{J}(\mathcal{A}'_{\mathrm{hf}}) \oplus \mathrm{J}(A').$$

as abelian groups. Note that $\mathrm{J}(A) = \mathrm{J}(A') = 0$ by Lemma 5.3. Thus we have

$$\mathrm{J}(\mathcal{A}_{\mathrm{hf}}) \cong \mathrm{J}(\mathcal{A}'_{\mathrm{hf}}).$$

as abelian groups. If $g \geq 6$ and g is even, then $\mathrm{J}(\mathcal{A}_{\mathrm{hf}}) \cong \mathrm{J}(\mathcal{K}u(Y))$ for some smooth del Pezzo threefold Y of degree $d = \frac{g-1}{2}$. Then we get the isomorphism of complex torus

$$\mathrm{J}(\mathcal{K}u(Y)) \cong \mathrm{J}(\mathcal{K}u(Y')),$$

since Y and Y' are smooth Fano variety and their Kuznetsov components are admissible subcategories of $D^b(Y)$ and $D^b(Y')$ respectively.

Now we show the isomorphism $J(\mathcal{K}u(Y)) \cong J(\mathcal{K}u(Y'))$ is an isomorphism of principle polarized abelian varieties. According to Lemma 5.3 and Lemma 3.7, we have an isometry preserving the Euler paring:

$$\mathrm{K}_1^{top}(\mathcal{K}u(X)_{\mathrm{hf}})_{tf} \cong \mathrm{K}_1^{top}(\mathcal{K}u(Y))_{tf}.$$

$$\mathrm{K}_1^{top}(\mathcal{K}u(X')_{\mathrm{hf}})_{tf} \cong \mathrm{K}_1^{top}(\mathcal{K}u(Y'))_{tf}.$$

On the other hand, the equivalence $\mathcal{K}u(X)_{\mathrm{hf}} \simeq \mathcal{K}u(X')_{\mathrm{hf}}$ gives an isometry preserving their Euler pairing:

$$\mathrm{K}_1^{top}(\mathcal{K}u(X)_{\mathrm{hf}})_{tf} \cong \mathrm{K}_1^{top}(\mathcal{K}u(X')_{\mathrm{hf}})_{tf}.$$

Thus we have a Hodge isometry

$$\mathrm{K}_1^{top}(\mathcal{K}u(Y))_{tf} \cong \mathrm{K}_1^{top}(\mathcal{K}u(Y'))_{tf}.$$

Therefore we have an isomorphism of polarised abelian varieties $\mathrm{J}(\mathcal{K}u(Y)) \cong \mathrm{J}(\mathcal{K}u(Y'))$.

Note that by Theorem 5.4, $\mathrm{J}(\mathcal{K}u(Y)) \cong J(Y) \cong J(X)$. Then we have $J(X) \cong J(X')$ as polarised abelian varieties. The other cases are argued similarly, we omit the details. □

Next we prove (birational) categorical Torelli theorem for nodal prime Fano threefolds.

Theorem 6.2. *Let X and X' be 1-nodal maximally non-factorial index one prime Fano threefold of genus $g \geq 6$ such that $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$, then $X \simeq X'$.*

Proof. We have $\mathcal{K}u(X)_{\mathrm{hf}} \simeq \mathcal{K}u(X')_{\mathrm{hf}}$. Apply intermediate Jacobian in Definition 5.1, by Theorem 5.4 we get

$$J(X) \cong \mathrm{J}(\mathcal{K}u(X)_{\mathrm{hf}}) \cong \mathrm{J}(\mathcal{K}u(X')_{\mathrm{hf}}) \cong J(X'),$$

as polarised abelian varieties, then $X \simeq X'$ from Theorem 4.6. □

Proposition 6.3. *Let X, X' be 1-nodal maximally non-factorial prime Fano threefold of index one and genus 7 with the assumption in Proposition 4.7, such that $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$, then $X \cong X'$.*

Proof. We have an equivalence $\mathcal{K}u(X)_{\mathrm{hf}} \simeq \mathcal{K}u(X')_{\mathrm{hf}}$, then we get $J(X) \cong J(X')$ as polarised abelian varieties. Thus the result follows from Proposition 4.7. □

As a corollary, we show that the bounded derived category $D^b(X)$ of coherent sheaves on these nodal prime Fano threefolds determines their birational isomorphism class.

Corollary 6.4. *Let X, X' be 1-nodal maximally non-factorial index one prime Fano threefolds of genus $g \geq 6$ such that $D^b(X) \simeq D^b(X')$, then $X \simeq X'$.*

Proof. By Theorem 4.5, there is a semi-orthogonal decomposition

$$D^b(X) = \langle \mathcal{K}u(X), \mathcal{O}_X, \mathcal{U}_X^\vee \rangle,$$

where $\mathcal{U}_X^\vee$ is the Mukai bundle of X. Then $D^b(X) \simeq D^b(X') \implies D^{\mathrm{perf}}(X) \simeq D^{\mathrm{perf}}(X') \implies \langle \mathcal{K}u(X)_{\mathrm{hf}}, \mathcal{O}_X, \mathcal{U}_X^\vee \rangle \simeq \langle \mathcal{K}u(X')_{\mathrm{hf}}, \mathcal{O}_{X'}, \mathcal{U}_{X'}^\vee \rangle$. Then apply intermediate Jacobian construction in Definition 5.1, we get $\mathrm{J}(\mathcal{K}u(X)_{\mathrm{hf}}) \cong \mathrm{J}(\mathcal{K}u(X')_{\mathrm{hf}})$, then we get $J(X) \cong J(X')$ as polarised abelian varieties. Thus we have $X \simeq X'$ from Theorem 4.6. □

6.2. **Derived Torelli theorem for nodal curves.** In this section, we prove Theorem 1.5 (1). Let $C_i = C_i' \bigcup C_i''$ be a reducible Gorenstein curve with $C_i' \cong \mathbb{P}^1$ and C_i'' is a smooth curve of genus $g(C_i'') > 1$ and $C_i' \bigcap C_i''$ is a single point x_i, which is smooth on C_i''.

Theorem 6.5. *Let C_1, C_2 be nodal curves as above, if $D^b(C_1) \simeq D^b(C_2)$, then $C_1 \cong C_2$.*

Proof. By absorption of singularities [KS23, Proposition 6.15], there is a semi-orthogonal decomposition

$$D^b(C_i) = \langle \mathcal{P}_i, \sigma_i^*(D^b(C_i'')) \rangle.$$

Thus

$$D^b(C_i)_{\mathrm{hf}} = \langle (\mathcal{P}_i)_{\mathrm{hf}}, D^b(C_i'') \rangle.$$

If

$$D^b(C_1) \simeq D^b(C_2),$$

then

$$D^b(C_1)_{\mathrm{hf}} \simeq D^b(C_2)_{\mathrm{hf}}.$$

Apply intermediate Jacobian in Definition 5.1, we get

$$\mathrm{J}(D^b(C_1)_{\mathrm{hf}}) \cong \mathrm{J}(D^b(C_2)_{\mathrm{hf}}).$$

Then we get $J(C_1'') \cong J(C_2'')$ as principal polarised abelian varieties. Then $C_1'' \cong C_2''$ by Torelli theorem for smooth curves of genus greater than one. On the other hand, $C_1' \cong C_2' \cong \mathbb{P}^1$, thus we get $C_1 \cong C_2$. □

7. Refined Categorical Torelli theorems for nodal prime Fano threefolds

In this section, we discuss *refined categorical Torelli problems* for 1-nodal maximally non-factorial prime Fano threefolds. Let X be a 1-nodal maximally non-factorial prime Fano threefold of genus 6 or 8. Then by Theorem 1.5, the Kuznetsov component $\mathcal{K}u(X)$ determines its birational isomorphism class. Thus it would be natural to ask what extra data we need to determine the isomorphism class of these nodal Fano threefolds.

For the case of smooth prime Fano threefolds of index one and genus $g \geq 6$, in the earlier paper [JLZ22], the authors start with the semi-orthogonal decomposition of such a prime Fano threefold X:

$$D^b(X) = \langle \mathcal{K}u(X), \mathcal{Q}_X^\vee, \mathcal{O}_X \rangle,$$

such that the Kuznetsov component is the right orthogonal complement of $\mathcal{O}_X$ and $\mathcal{Q}_X^\vee$. Denote by $i : \mathcal{K}u(X) \hookrightarrow \langle \mathcal{K}u(X), \mathcal{Q}_X^\vee \rangle$ the inclusion functor. Then they show the isomorphism class of X is uniquely determined by the category $\mathcal{K}u(X)$ together with a distinguished object $i^!\mathcal{Q}_X^\vee$. The idea of the proof is to look at how the Kuznetsov component and the distinguished object produce a classical invariant which is used to reconstruct X. For example, in the case of smooth Gushel-Mukai threefold($g = 6$), one can reconstruct the minimal model $\mathcal{C}_m(X)$ of Fano surface of conics on X from $\mathcal{K}u(X)$ via Bridgeland moduli space, and the distinguished object $i^!\mathcal{Q}_X^\vee$ represents a special point on $\mathcal{C}_m(X)$ whose blow up is the honest Fano surface $C(X)$ of conics on X, which determines the isomorphism class of X, by a result of Logchev [Log12].

For the case of 1-nodal maximally non-factorial prime Fano threefold X of index one and genus $g = 2d + 2$, we also consider its semi-orthogonal decomposition

$$D^b(X) = \langle \mathcal{K}u(X), \mathcal{O}_X, \mathcal{U}_X^\vee \rangle,$$

where $\mathcal{U}_X$ is the Mukai bundle, which produces the embedding of X into Grassmannian $\mathrm{Gr}(2, d+3)$. Moreover, by [KS23, Proposition 3.3], the Kuznetsov component $\mathcal{K}u(X)$ has further decomposition

$$\mathcal{K}u(X) = \langle \mathcal{P}, \mathcal{A}_X \rangle,$$

where the category $\mathcal{P}$ is generated by a $\mathbb{P}^{\infty,2}$ object and $\mathcal{A}_X$ is a smooth proper category, which is equivalent to Kuznetsov component $\mathcal{K}u(Y)$ of a smooth del Pezzo threefold of degree d. Denote $\mathbf{L}_{\mathcal{O}_X}\mathcal{U}_X^\vee[-1]$ by $\mathcal{Q}_X^\vee$. It is a vector bundle if $d \geq 3$ since $\mathcal{U}_X^\vee$ is a globally generated vector bundle. Still denote by $i : \mathcal{K}u(X) \hookrightarrow \langle \mathcal{K}u(X), \mathcal{Q}_X^\vee \rangle$ the inclusion functor. The distinguished object for X should be defined as $\mathbf{R}_{\mathcal{P}}i^!\mathcal{Q}_X^\vee$. On the other hand, by classification [CKGS23] of 1-nodal maximally non-factorial prime Fano threefold, such a degree $2d+2$ 1-nodal maximally non-factorial prime Fano threefold is constructed from degree d del Pezzo threefold via *Bridge construction*. For example, 1-nodal maximally non-factorial Gushel-Mukai threefold X is uniquely determined by a smooth quartic double solid Y and a line $L \subset Y$. Thus to show $\mathbf{R}_{\mathcal{P}}i^!\mathcal{Q}_X^\vee$ determines the isomorphism class of X, one should be able to produce a line on Y from this object. Similarly, for genus $g = 8$ case, one should relate the object $\mathbf{R}_{\mathcal{P}}i^!\mathcal{Q}_X^\vee$ to a conic on the cubic threefold we start with.

Let $\widetilde{Y}$ be the small resolution of X, whose semi-orthogonal decomposition is given in the proof [KS23, Proposition 3.3] as

$$D^b(\widetilde{Y}) = \langle \mathcal{O}_{\widetilde{Y}}(E-H), \mathbf{T}_{\mathcal{O}_{\widetilde{L}}(-1)}(\mathcal{O}_{\widetilde{Y}}(E-H)), \mathbf{R}_{\mathbf{T}_{\mathcal{O}_{\widetilde{L}(-1)}}(\mathcal{O}_{\widetilde{Y}}(E-H))}(\widetilde{\mathcal{B}}_Y), \mathcal{O}_{\widetilde{Y}}, \mathcal{U}^\vee_{\widetilde{Y}}\rangle,$$

where $\mathbf{T}_{\mathcal{O}_{\widetilde{L}}(-1)}(\mathcal{O}_{\widetilde{Y}}(E-H))$ is the spherical twist of $\mathcal{O}_{\widetilde{Y}}(E-H)$ via $\mathcal{O}_{\widetilde{L}}(-1)$, which fits the exact triangle

$$\mathcal{O}_{\widetilde{L}}(-1)[-2] \to \mathcal{O}_{\widetilde{Y}}(E-H) \to \mathbf{T}_{\mathcal{O}_{\widetilde{L}}(-1)}(\mathcal{O}_{\widetilde{Y}}(E-H)).$$

Denote by $\mathbf{T}$ the object $\mathbf{T}_{\mathcal{O}_{\widetilde{L}}(-1)}(\mathcal{O}_{\widetilde{Y}}(E-H))$. Note that $\mathbf{R}_{\mathbf{T}_{\mathcal{O}_{\widetilde{L}}(-1)}(\mathcal{O}_{\widetilde{Y}}(E-H))}(\widetilde{\mathcal{B}}_Y) \simeq \mathcal{K}u(Y)$, via an equivalence $\sigma_* \circ \mathbf{L}_{\mathbf{T}}$. Then we set $\mathcal{K}u(\widetilde{Y}) := \langle \mathcal{O}_{\widetilde{Y}}(E-H), \sigma^*\mathcal{K}u(Y), \mathbf{T}\rangle = \langle \mathcal{O}_{\widetilde{Y}}, \mathcal{U}^\vee_{\widetilde{Y}}\rangle^\perp$, where $\sigma : \widetilde{Y} \to Y$ is the blow up of Y along a smooth curve C of degree $d-1$. Set

$$\mathcal{D} = \langle \mathcal{K}u(\widetilde{Y}), \mathcal{Q}^\vee_{\widetilde{Y}}\rangle,$$

where $\mathcal{Q}^\vee_{\widetilde{Y}} \cong \mathbf{L}_{\mathcal{O}_{\widetilde{Y}}}\mathcal{U}^\vee_{\widetilde{Y}}[-1]$ is a rank $d+1$ vector bundle whenever $d \geq 3$. Denote by $i : \mathcal{K}u(\widetilde{Y}) \hookrightarrow \mathcal{D}$ the inclusion functor, then we produce an object $\mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} \circ \mathbf{L}_{\mathbf{T}} i^! \mathcal{Q}^\vee_{\widetilde{Y}} \in \sigma^*\mathcal{K}u(Y)$, where (σ^*, σ_*) induces an equivalence between $\mathcal{K}u(Y)$ and $\widetilde{\mathcal{B}}_Y$. The next lemma expresses the object $i^!\mathcal{Q}^\vee_{Y}$ as a two-term complex, so that we can compute $\mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} \circ \mathbf{L}_{\mathbf{T}} i^! \mathcal{Q}^\vee_{\widetilde{Y}}$.

Lemma 7.1.

(1) $\mathrm{RHom}(\mathcal{Q}^\vee_{\widetilde{Y}}, \mathcal{O}_{\widetilde{Y}}(E-H)) = 0$.
(2) $\mathrm{RHom}(\mathcal{Q}^\vee_{\widetilde{Y}}, \mathbf{T}) = 0$.
(3) $\mathrm{RHom}(\mathcal{U}_{\widetilde{Y}}, \mathcal{O}_{\widetilde{Y}}(E-H)) = k^2$.
(4) $\mathrm{RHom}(\mathcal{O}_E(E-F), \mathbf{T}) = k^2[-2]$.

Proof. The first two equality simply follows from the proof in [Kuz04b, Proposition 3.3]. Indeed, $\mathcal{Q}^\vee_{\widetilde{Y}} \in {}^\perp\langle \mathcal{O}_{\widetilde{Y}(E-H)}, \mathbf{T}\rangle$. Thus the result follows. As $\mathrm{RHom}(\mathcal{U}_{\widetilde{Y}}, \mathcal{O}_{\widetilde{Y}}(E-H)) = \mathrm{RHom}(\mathcal{O}_{\widetilde{Y}}, \mathcal{U}^\vee_{\widetilde{Y}}(E-H))$. Note that there is a short exact sequence [KS23, Lemma 2.13]:

$$0 \to \mathcal{U}^\vee_{\widetilde{Y}} \to \mathcal{O}(H)^{\oplus 2} \to \mathcal{O}_F(3F) \to 0.$$

Twisted by a line bundle $\mathcal{O}_{\widetilde{Y}}(E-H)$, we get

$$0 \to \mathcal{U}^\vee_{\widetilde{Y}}(E-H) \to \mathcal{O}_{\widetilde{Y}}(E)^{\oplus 2} \to \mathcal{O}_F(E+F) \to 0.$$

Apply $\mathrm{Hom}(\mathcal{O}_{\widetilde{Y}}, -)$, we get a long exact sequence

$$0 \to \mathrm{H}^\bullet(\widetilde{Y}, \mathcal{U}^\vee_{\widetilde{Y}}(E-H)) \to \mathrm{H}^\bullet(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}(E)) \to 0,$$

Indeed, $\mathrm{H}^\bullet(\widetilde{Y}, \mathcal{O}_E(E+F)) \cong \mathrm{H}^\bullet(\rho^*\mathcal{O}_C, \mathcal{O}_E(-1) \otimes \rho^*\mathcal{O}_C(1)) \cong \mathrm{H}^\bullet(C, \rho_*\mathcal{O}_E(-1) \otimes \mathcal{O}_C(1)) = 0$ since the rank of pushforward $\rho_*\mathcal{O}_E(-1)$ is given by $h^0(F, \mathcal{O}_F(-1)) = 0$. Thus $\mathrm{RHom}(\mathcal{U}_{\widetilde{Y}}, \mathcal{O}(E-H) = k^2$. Then (3) follows. Before proving (4), we first compute the right mutation $\mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)}\mathcal{U}_{\widetilde{Y}}$, which is given by the exact triangle

$$\mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)}\mathcal{U}_{\widetilde{Y}} \to \mathcal{U}_{\widetilde{Y}} \to \mathcal{O}_{\widetilde{Y}}(E-H) \otimes \mathrm{RHom}(\mathcal{U}_{\widetilde{Y}}, \mathcal{O}_{\widetilde{Y}}(E-H))^*.$$

then by (1), we get the triangle $\mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)}\mathcal{U}_{\widetilde{Y}} \to \mathcal{U}_{\widetilde{Y}} \to \mathcal{O}_{\widetilde{Y}}(E-H)^{\oplus 2}$. But we also have a short exact sequence

$$0 \to \mathcal{U}^\vee_{\widetilde{Y}}(K_{\widetilde{Y}}) \to \mathcal{O}_{\widetilde{Y}}(E-H)^{\oplus 2} \to \mathcal{O}_E(E-F) \to 0.$$

Then we have $\mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)}\mathcal{U}_{\widetilde{Y}} \cong \mathcal{O}_E(E-F)[-1]$. Then we immediately have $\mathrm{RHom}(\mathcal{O}_E(E-F), \mathcal{O}_{\widetilde{Y}}(E-H)) = 0$. Note that $\mathbf{T}$ is given by the triangle

$$\mathcal{O}_{\widetilde{Y}}(E-H) \to \mathbf{T} \to \mathcal{O}_{\widetilde{L}}(-1)[-1].$$

Then $\mathrm{RHom}(\mathcal{O}_E(E-F), \mathbf{T}) \cong \mathrm{RHom}(\mathcal{O}_E(E-F), \mathcal{O}_{\widetilde{L}}(-1)[-1])$. Apply $\mathrm{RHom}(-, \mathcal{O}_{\widetilde{L}}(-1)[-1])$ to the short exact sequence

$$0 \to \mathcal{U}^\vee_{\widetilde{Y}}(K_{\widetilde{Y}}) \to \mathcal{O}_{\widetilde{Y}}(E-H)^{\oplus 2} \to \mathcal{O}_E(E-F) \to 0,$$

Note that $\mathcal{U}^\vee_{\widetilde{Y}}(K_{\widetilde{Y}}) \cong \mathcal{U}_{\widetilde{Y}}$ restricting $\widetilde{L}$ is trivial, thus $\mathrm{RHom}(\mathcal{U}_{\widetilde{Y}}, \mathcal{O}_{\widetilde{L}}(-1)[-1]) = 0$. Then we have $\mathrm{RHom}(\mathcal{O}_E(E-F), \mathcal{O}_{\widetilde{L}}(-1)[-1]) \cong \mathrm{RHom}(\mathcal{O}_{\widetilde{Y}}(E-H)^{\oplus 2}, \mathcal{O}_{\widetilde{L}}(-1)[-1]) = \mathrm{H}^\bullet(\mathcal{O}_{\widetilde{L}}(-2))^{\oplus 2}[-1] \cong k^2[-2]$. Then (4) follows. □

Lemma 7.2.

(1) The distinguished object $i^!\mathcal{Q}^\vee_{\widetilde{Y}} \cong \mathbf{L}_{\mathcal{Q}^\vee_{\widetilde{Y}}}\mathcal{U}_{\widetilde{Y}}[1] \cong \mathbf{L}_{\mathcal{Q}^\vee_{\widetilde{Y}}}\mathcal{U}^\vee_{\widetilde{Y}}(K_{\widetilde{Y}})[1]$.

(2) It fits the exact triangle

$$\mathcal{U}_{\widetilde{Y}}[1] \to ii^!\mathcal{Q}^\vee_{\widetilde{Y}} \to \mathcal{Q}^\vee_{\widetilde{Y}}.$$

Proof. The argument is almost the same as [JLZ22, Lemma 3.4]. First we have exact triangle

$$ii^!\mathcal{Q}^\vee_{\widetilde{Y}} \to \mathcal{Q}^\vee_{\widetilde{Y}} \to \mathbf{L}_{\mathcal{K}u(\widetilde{Y})}\mathcal{Q}^\vee_{\widetilde{Y}}.$$

Note that $\langle \mathcal{K}u(\widetilde{Y}), \mathcal{Q}^\vee_{\widetilde{Y}} \rangle = \langle \mathcal{S}_{\mathcal{D}}\mathcal{Q}^\vee_{\widetilde{Y}}, \mathcal{K}u(\widetilde{Y}) \rangle = \langle \mathbf{L}_{\mathcal{K}u(\widetilde{Y})}\mathcal{Q}^\vee_{\widetilde{Y}}, \mathcal{K}u(\widetilde{Y}) \rangle$, where $\mathcal{S}_{\mathcal{D}}$ is the Serre functor of $\mathcal{D}$. Then we get the triangle

$$ii^!\mathcal{Q}^\vee_{\widetilde{Y}} \to \mathcal{Q}^\vee_{\widetilde{Y}} \to \mathcal{S}_{\mathcal{D}}\mathcal{Q}^\vee_{\widetilde{Y}}.$$

But it is not hard to see $\mathcal{S}_{\mathcal{D}}\mathcal{Q}^\vee_{\widetilde{Y}} \cong \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(K_{\widetilde{Y}})}(\mathcal{Q}^\vee_{\widetilde{Y}}(K_{\widetilde{Y}})[3]) \cong \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}}\mathcal{Q}^\vee_{\widetilde{Y}} \otimes \mathcal{O}_{\widetilde{Y}}(K_{\widetilde{Y}})[3] \cong \mathcal{U}^\vee_{\widetilde{Y}}(K_{\widetilde{Y}})[2]$. Then we got the exact triangle

$$\mathcal{U}^\vee_{\widetilde{Y}}(K_{\widetilde{Y}})[1] \to ii^!\mathcal{Q}^\vee_{\widetilde{Y}} \to \mathcal{Q}^\vee_{\widetilde{Y}}.$$

Note that $\mathcal{U}^\vee_{\widetilde{Y}}(K_{\widetilde{Y}}) \cong \mathcal{U}_{\widetilde{Y}}$ since $\mathcal{U}_{\widetilde{Y}}$ is a rank two bundle and determinant of $\mathcal{U}_{\widetilde{Y}}$ and $\mathcal{U}^\vee_{\widetilde{Y}}(K_{\widetilde{Y}})$ are the same. Apply the left adjoint functor $i^*_{\mathcal{D}}$ of i, we get $i^!\mathcal{Q}^\vee_{\widetilde{Y}} \cong \mathbf{L}_{\mathcal{Q}^\vee_{\widetilde{Y}}}\mathcal{U}_{\widetilde{Y}}[1]$. □

Lemma 7.3. *The left mutation* $\mathbf{L}_{\mathbf{T}} \circ \mathbf{R}_{\mathbf{T}}\, \mathcal{O}_E(E-F) \cong \mathcal{O}_E(E-F)$ *and* $\mathbf{L}_{\mathbf{T}}\mathcal{Q}^\vee_{\widetilde{Y}}$ *fits into the triangle*

$$\mathbf{T}^2 \to \mathcal{Q}^\vee_{\widetilde{Y}} \to \mathbf{L}_{\mathbf{T}}\mathcal{Q}^\vee_{\widetilde{Y}}.$$

Proof. There is a triangle

$$\mathbf{R}_{\mathbf{T}}\, \mathcal{O}_E(E-F) \to \mathcal{O}_E(E-F) \to \mathbf{T}^{\oplus 2}[2].$$

Apply $\mathbf{L}_{\mathbf{T}}$, we get the triangle $\mathbf{L}_{\mathbf{T}} \circ \mathbf{R}_{\mathbf{T}}\, \mathcal{O}_E(E-F) \to \mathbf{L}_{\mathbf{T}}\, \mathcal{O}_E(E-F) \to 0$, thus $\mathbf{L}_{\mathbf{T}} \circ \mathbf{R}_{\mathbf{T}}\, \mathcal{O}_E(E-F) \cong \mathbf{L}_{\mathbf{T}}\, \mathcal{O}_E(E-F)$. We have a short exact sequence

$$0 \to \mathcal{U}_{\widetilde{Y}} \to \mathcal{O}(E-H)^{\oplus 2} \to \mathcal{O}_E(E-F) \to 0.$$

Then apply $\mathrm{RHom}(\mathbf{T}, -)$ we get

$$0 \to \mathrm{RHom}(\mathbf{T}, \mathcal{U}_{\widetilde{Y}}) \to \mathrm{RHom}(\mathbf{T}, \mathcal{O}_{\widetilde{Y}}(E-H)) \to \mathrm{RHom}(\mathbf{T}, \mathcal{O}_E(E-F)),$$

where $\mathrm{RHom}(\mathbf{T}, \mathcal{U}_{\widetilde{Y}}) = 0$ and $\mathrm{RHom}(\mathbf{T}, \mathcal{O}_{\widetilde{Y}}(E-H)) = 0$ since there is a semi-orthogonal decomposition

$$D^b(\widetilde{Y}) = \langle \mathcal{O}_{\widetilde{Y}}(E-H), \mathbf{T}_{\mathcal{O}_{\widetilde{L}}(-1)}(\mathcal{O}_{\widetilde{Y}}(E-H)), \mathbf{R}_{\mathbf{T}_{\mathcal{O}_{\widetilde{L}(-1)}}(\mathcal{O}_{\widetilde{Y}}(E-H))}(\widetilde{\mathcal{B}}_Y), \mathcal{O}_{\widetilde{Y}}, \mathcal{U}^\vee_{\widetilde{Y}} \rangle.$$

Then $\mathrm{RHom}(\mathbf{T}, \mathcal{O}_E(E-F)) = 0$, thus $\mathbf{L_T}\,\mathcal{O}_E(E-F) \cong \mathcal{O}_E(E-F)$. Next we compute $\mathrm{RHom}(\mathbf{T}, \mathcal{Q}^\vee_{\widetilde{Y}})$. Apply $\mathrm{Hom}(-, \mathcal{Q}^\vee_{\widetilde{Y}})$ to the triangle

$$\mathbf{T} \to \mathcal{O}_{\widetilde{Y}}^{\oplus 2} \to \mathcal{O}_{\widetilde{Y}}(H-E).$$

Then we get $\mathrm{Hom}^\bullet(\mathbf{T}, \mathcal{Q}^\vee_{\widetilde{Y}}) \cong \mathrm{Hom}^{\bullet+1}(\mathcal{O}_{\widetilde{Y}}(H-E), \mathcal{Q}^\vee_{\widetilde{Y}})$. Apply $\mathrm{Hom}(\mathcal{O}_{\widetilde{Y}}(H-E), -)$ to the short exact sequence

$$0 \to \mathcal{Q}^\vee_{\widetilde{Y}} \to \mathcal{O}_{\widetilde{Y}}^{\oplus 6} \to \mathcal{U}^\vee_{\widetilde{Y}} \to 0.$$

We get $\mathrm{Hom}^\bullet(\mathcal{O}_{\widetilde{Y}}(H-E), \mathcal{U}^\vee_{\widetilde{Y}}) \cong \mathrm{Ext}^{\bullet+1}(\mathcal{O}_{\widetilde{Y}}(H-E), \mathcal{Q}^\vee_{\widetilde{Y}})$. Thus $\mathrm{Hom}^\bullet(\mathbf{T}, \mathcal{Q}^\vee_{\widetilde{Y}}) \cong \mathrm{Hom}^\bullet(\mathcal{O}_{\widetilde{Y}}(H-E), \mathcal{U}^\vee_{\widetilde{Y}}) \cong \mathrm{Hom}^\bullet(\mathcal{U}_{\widetilde{Y}}, \mathcal{O}_{\widetilde{Y}}(E-H)) = k^2$ by Lemma 7.1. Then the desired triangle for $\mathbf{L_T}\mathcal{Q}^\vee_{\widetilde{Y}}$ is obtained.

□

Proposition 7.4. *Let Y be a del Pezzo threefold of degree $d \geq 2$ and $\sigma : \widetilde{Y} \to Y$ be the blow up of a smooth rational curve $C \subset Y$ of degree $d-1$, then*

(1) $\sigma_*\mathbf{T}$ fits into exact triangle

$$\mathcal{O}_Y(-H)[1] \to \sigma_*\mathbf{T}[1] \to \mathcal{O}_L(-1),$$

Thus $\sigma_*\mathbf{T} \cong \mathcal{J}_L[-1]$, where $\mathcal{J}_L$ is the twisted derived dual of ideal sheaf of the line L.

(2) $\sigma_*\,\mathcal{O}_E(E-F) = 0$.

(3) $\sigma_*\mathcal{U}^\vee_{\widetilde{Y}} \cong E \otimes \mathcal{O}_Y(H)$, where E is a non-locally free sheaf that fits the short exact sequence

$$0 \to E \to \mathcal{O}_Y^{\oplus 2} \to \mathcal{O}_C(p) \to 0.$$

Proof. First note that we have an exact triangle

$$\mathbf{T} \to \mathcal{O}_{\widetilde{Y}}^{\oplus 2} \to \mathcal{O}_{\widetilde{Y}}(H-E).$$

Then we get a triangle

$$\sigma_*\mathbf{T} \to \mathcal{O}_Y^{\oplus 2} \xrightarrow{\xi} \mathcal{O}_Y(H) \otimes I_C,$$

where C is a smooth rational curve of degree $d-1$ on Y. Note that the map ξ is not surjective and the image of ξ is $\mathcal{O}_Y(H) \otimes I_D$, where D is the codimension two linear section, which is a degree d elliptic curve containing C and the residual curve of C in D is a line $L \subset Y$. Taking cohomology with respect to the standard heart, we get a long exact sequence

$$0 \to \mathcal{H}^0(\sigma_*\mathbf{T}) \to \mathcal{O}_Y^{\oplus 2} \to \mathcal{O}_Y(H) \otimes I_C \to \mathcal{H}^1(\sigma_*\mathbf{T}) \to 0.$$

Then we get two short exact sequence

$$0 \to \mathcal{H}^0(\sigma_*\mathbf{T}) \to \mathcal{O}_Y^{\oplus 2} \to I_D \otimes \mathcal{O}_Y(H) \to 0,$$

and

$$0 \to \mathcal{O}_Y(H) \otimes I_D \to \mathcal{O}_Y(H) \otimes I_C \to \mathcal{H}^1(\sigma_*\mathbf{T}) \to 0.$$

It is clear that $\mathcal{H}^0(\sigma_*\mathbf{T}) \cong \mathcal{O}_Y(-H)$ and $\mathcal{H}^1(\sigma_*\mathbf{T}) \cong \mathcal{O}_L(-1)$. Then we get

$$\mathcal{O}_Y(-H)[1] \to \sigma_*\mathbf{T}[1] \to \mathcal{O}_L(-1),$$

Then $\sigma_*\mathbf{T}[1] \cong \mathcal{J}_L$, thus $\sigma_*\mathbf{T} \cong \mathcal{J}_L[-1]$, which proves (1). We apply the Grothendieck-Riemann-Roch formula to compute the character of $\sigma_*\,\mathcal{O}_E(E-F)$, which is 0 and note that $\mathrm{R}^i\,\mathcal{O}_E(E-F) \otimes \mathcal{O}_y \cong \mathrm{H}^i(F, \mathcal{O}_F(-1)) = 0$ for all $i \geq 0$, thus $\sigma_*\,\mathcal{O}_E(E-F) = 0$, and this proves (2). Next, note that there is a short exact sequence

$$0 \to \mathcal{U}^\vee_{\widetilde{Y}} \to \mathcal{O}_{\widetilde{Y}}(H)^{\oplus 2} \to \mathcal{O}_E(dF) \to 0.$$

Apply σ_* to the sequence above we get a short exact sequence

$$0 \to \sigma_*\mathcal{U}^\vee_{\widetilde{Y}} \to \mathcal{O}_Y(H)^{\oplus 2} \to \mathcal{O}_C(d) \to 0,$$

since $\mathrm{R}^1\sigma_*\mathcal{U}^\vee_{\widetilde{Y}} = 0$. Then we twist the short exact sequence obtained by the line bundle $\mathcal{O}_Y(-H)$, we get a short exact sequence

$$0 \to \sigma_*\mathcal{U}^\vee_{\widetilde{Y}} \otimes \mathcal{O}_Y(-H) \to \mathcal{O}_Y^{\oplus 2} \to \mathcal{O}_C(1) \to 0.$$

Then $\sigma_*\mathcal{U}^\vee_{\widetilde{Y}} \otimes \mathcal{O}_Y(-H) \cong E$, where E is the non-locally free sheaf associated with the smooth rational curve C of degree $d-1$. Then $\sigma_*\mathcal{U}^\vee_{\widetilde{Y}} \cong E \otimes \mathcal{O}_Y(H)$, which proves (3). $\square$

Now we compute the object $\mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} \circ \mathbf{L_T} i^! \mathcal{Q}^\vee_{\widetilde{Y}}$. We omit the subscript if there is no confusion. Then we apply $\mathbf{R}_{\mathcal{O}(E-H)} \circ \mathbf{L_T}$ to the triangle $\mathcal{U}[1] \to ii^!\mathcal{Q}^\vee \to \mathcal{Q}^\vee$. We get

$$\mathbf{L_T}\mathcal{U}_{\widetilde{Y}}[1] \to \mathbf{L_T} ii^! \mathcal{Q}^\vee_{\widetilde{Y}} \to \mathbf{L_T}\mathcal{Q}^\vee_{\widetilde{Y}},$$

where $\mathbf{L_T}\mathcal{U}_{\widetilde{Y}} \cong \mathcal{U}_{\widetilde{Y}}$ since $\mathrm{Hom}(\mathbf{T}, \mathcal{U}_{\widetilde{Y}}) = 0$. Then we apply $\mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)}$ on $\mathcal{U}_{\widetilde{Y}}[1]$, we get $\mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)}\mathcal{U}_{\widetilde{Y}}[1] \cong \mathcal{O}_E(E-F)$. On the other hand, $\mathbf{L_T}\mathcal{Q}^\vee_{\widetilde{Y}}$ is given by the triangle

$$\mathbf{T}^2 \to \mathcal{Q}^\vee_{\widetilde{Y}} \to \mathbf{L_T}\mathcal{Q}^\vee_{\widetilde{Y}}.$$

Then we apply $\mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)}$ to the triangle above. Since $\mathrm{Hom}(\mathbf{T}, \mathcal{O}_{\widetilde{Y}}(E-H)) = \mathrm{Hom}(\mathcal{Q}^\vee_{\widetilde{Y}}, \mathcal{O}_{\widetilde{Y}}(E-H)) = 0$, thus we have the triangle

$$\mathbf{T}^2 \to \mathcal{Q}^\vee_{\widetilde{Y}} \to \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} \circ \mathbf{L_T}\mathcal{Q}^\vee_{\widetilde{Y}}.$$

Then we get a triangle

$$\mathcal{O}_E(E-F) \to \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} \circ \mathbf{L_T} ii^! \mathcal{Q}^\vee_{\widetilde{Y}} \to \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} \circ \mathbf{L_T}\mathcal{Q}^\vee_{\widetilde{Y}}.$$

Finally, we apply the functor σ_* to push forward the object above into $\mathcal{K}u(Y)$. Then we get

$$\sigma_* \circ \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} \circ \mathbf{L_T} ii^! \mathcal{Q}^\vee_{\widetilde{Y}} \cong \sigma_* \circ \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} \circ \mathbf{L_T}\mathcal{Q}^\vee_{\widetilde{Y}},$$

since $\sigma_*\mathcal{O}_E(E-F) = 0$ by Proposition 7.4(2). On the other hand, by Lemma 7.3, the object $\mathbf{L_T}\mathcal{Q}^\vee_{\widetilde{Y}}$ fits into the exact triangle

$$\mathbf{T}^2 \to \mathcal{Q}^\vee_{\widetilde{Y}} \to \mathbf{L_T}\mathcal{Q}^\vee_{\widetilde{Y}}.$$

Definition 7.5. We define the *gluing object* $\mathcal{G}$ as follows

$$\mathcal{G} = \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} \circ \mathbf{L_T}\mathcal{Q}^\vee_{\widetilde{Y}} \in \sigma^*(Ku(Y)).$$

Then $\mathcal{G}$ fits into the triangle

$$\sigma_*\mathbf{T}^2 \to \sigma_*\mathcal{Q}^\vee_{\widetilde{Y}} \to \sigma_*(\mathcal{G}).$$

Note that by Proposition 7.4, $\sigma_*\mathbf{T}[1] \cong \mathcal{J}_L^{\oplus 2}$ and $\sigma_*\mathcal{Q}^\vee_{\widetilde{Y}}$ fits into the short exact sequence

$$0 \to \sigma_*\mathcal{Q}^\vee_{\widetilde{Y}} \to \mathcal{O}_Y^{\oplus(d+3)} \to E \otimes \mathcal{O}_Y(H) \to 0.$$

Next we identify the distinguished object $\sigma_*(\mathcal{G})$ with the (acyclic extension of) non-locally free instanton sheaf on smooth del Pezzo threefold Y of degree d associated to the smooth rational curve C of degree $d-1$ we start with.

7.1. **Bridge construction from Del Pezzo threefolds of degree $d \geq 3$.** Let $d \in \{2,3,4,5\}$. Let X be a 1-nodal maximally non-factorial prime Fano threefold of genus $2d+2$ obtained via the *bridge construction* from a smooth Del Pezzo threefold $Y := Y_d$ and a smooth rational curve $C \subset Y$ of degree $d-1$.

From C we obtain two sheaves E and F, which both fail to be locally free along C. Letting p be a point of C, the sheaves E and F are described as kernels of the evaluation maps as follows of $\mathcal{O}_C(p)$ and $\mathcal{O}_C((d-2)p)$, so we have exact sequences

$$0 \to E \to \mathcal{O}_Y^{\oplus 2} \to \mathcal{O}_C(p) \to 0, \tag{2}$$

$$0 \to F \to \mathcal{O}_Y^{\oplus(d-1)} \to \mathcal{O}_C((d-2)p) \to 0. \tag{3}$$

In other words $E = \mathbf{L}_{\mathcal{O}_Y} \mathcal{O}_C(p)[-1]$ and $F = \mathbf{L}_{\mathcal{O}_Y} \mathcal{O}_C((d-2)p)[-1] \cong \mathbf{L}_{\mathcal{O}_Y} F$.

For $d = 2$ we have $F = \mathcal{I}_C$ and for $d \geq 3$ and for any base-point-free pencil of sections of $\mathcal{O}_C((d-2)p)$, we get an instanton sheaf F_0 of rank 2 on Y fitting into

$$0 \to F_0 \to \mathcal{O}_Y^{\oplus 2} \to \mathcal{O}_C((d-2)p) \to 0.$$

Then F is the acyclic extension of F_0 in the sense of [Kuz12, Definition 1.1], fitting into

$$0 \to F_0 \to F \to \mathcal{O}_Y^{\oplus(d-3)} \to 0. \tag{4}$$

We have $H^*(F_0(-1)) = 0$ and $H^*(F(-1)) = H^*(F) = 0$ so F lies in $\mathcal{K}u(Y)$.

Theorem 7.6. *Let X be a 1-nodal maximally non-factorial prime Fano threefold of degree $2d+2$ obtained via the* bridge construction *from a smooth Del Pezzo threefold Y of degree $d \geq 2$ and a smooth rational curve $C \subset Y$ of degree $d-1$. Consider the gluing object $\sigma_*(\mathcal{G}) \in \mathcal{K}u(Y)$. Then*

$$\sigma_*(\mathcal{G}) \simeq F.$$

Proof of Theorem 7.6 for $d \geq 3$. Recall that, given the smooth curve $C \subset Y$, L is the residual line of C with respect to pencil of hyperplane sections of Y containing C. By definition, the gluing object $\mathcal{G}$ is

$$\mathcal{G} = \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} \circ \mathbf{L}_{\mathbf{T}} i^! \mathcal{Q}_{\widetilde{Y}}^\vee \in \sigma^*(\mathcal{K}u(Y)).$$

We note that $\sigma_*(\mathcal{G})$ fits into a canonical exact sequence

$$0 \to \mathcal{O}_Y(-1)^{\oplus 2} \to \sigma_*(\mathcal{Q}_{\widetilde{Y}}^\vee) \to \sigma_*(\mathcal{G}) \to \mathcal{O}_L(-1)^{\oplus 2} \to 0, \tag{5}$$

while $\sigma_*(\mathcal{Q}_{\widetilde{Y}}^\vee)$ fits into

$$0 \to \sigma_*(\mathcal{Q}_{\widetilde{Y}}^\vee) \to \mathcal{O}_Y^{\oplus(d+3)} \to E(1) \to 0. \tag{6}$$

Indeed, to check (5), by Proposition 7.4, we write the exact sequence

$$0 \to \mathcal{H}^{-1}(\sigma_*(\mathcal{G})) \to \mathcal{O}_Y(-1)^{\oplus 2} \to \sigma_*(\mathcal{Q}_{\widetilde{Y}}^\vee) \to \mathcal{H}^0(\sigma_*(\mathcal{G})) \to \mathcal{O}_L(-1)^{\oplus 2} \to 0.$$

Hence it suffices to check that $\mathcal{H}^{-1}(\sigma_*(\mathcal{G})) = 0$ and actually showing that $\mathrm{rk}(\mathcal{H}^{-1}(\sigma_*(\mathcal{G}))) = 0$ is enough as this sheaf is torsion-free. The map $f : \mathcal{O}_Y(-1)^{\oplus 2} \to \sigma_*(\mathcal{Q}_{\widetilde{Y}}^\vee)$ is dual to

$$g : \sigma_*(\mathcal{Q}_{\widetilde{Y}}^\vee)^\vee \to \mathcal{I}_{L/Y}(1)^{\oplus 2},$$

hence $\mathrm{rk}(\mathrm{im}(f)) = \mathrm{rk}(\mathrm{im}(g))$ and we have to verify $\mathrm{rk}(\mathrm{im}(g)) = 2$. Since the map g comes from evaluation of morphisms toward $\mathcal{I}_{L/Y}(1)$, we must have $\mathrm{Hom}(\mathrm{im}(g), \mathcal{I}_{L/Y}(1)) = k^2$. So, by stability of $\mathcal{I}_{L/Y}$ and $\sigma_*(\mathcal{Q}_{\widetilde{Y}}^\vee)$, if we had $\mathrm{rk}(\mathrm{im}(g)) < 2$ then $\mathrm{im}(g)$ would be isomorphic to $\mathcal{I}_{Z,Y}(1)$ for some subscheme Z of Y of codimension at least 2, with $L \subset Z$. But then we would get $\mathrm{Hom}(\mathrm{im}(g), \mathcal{I}_{L/Y}) = k$, a contradiction.

So (5) is proved. Our main task now it to show that F fits into a canonical exact sequence

$$0 \to \mathcal{O}_Y(-1)^{\oplus 2} \to \sigma_*(\mathcal{Q}^\vee_{\widetilde{Y}}) \to F \to \mathcal{O}_L(-1)^{\oplus 2} \to 0,$$

which has the same form as (5). Since these sequences are canonically attached to C, we deduce that $F \simeq \sigma_*(\mathcal{G})$. First, recall that L intersects C at a subscheme Z of length 2. Observe that $\mathcal{T}or_1(\mathcal{O}_C, \mathcal{O}_L) \simeq \mathcal{O}_Z$ so that, restricting (3) to L, we get

$$0 \to \mathcal{O}_Z \to F|_L \to \mathcal{O}_L^{\oplus(d-1)} \to \mathcal{O}_Z \to 0.$$

This implies

$$F|_L \simeq \mathcal{O}_Z \oplus \mathcal{O}_L(-1)^{\oplus 2} \oplus \mathcal{O}_L^{\oplus(d-3)}.$$

We define $F_1 = \ker(F \to \mathcal{O}_L(-1)^{\oplus 2})$, the map here being the obvious surjection, hence:

$$0 \to F_1 \to F \to \mathcal{O}_L(-1)^{\oplus 2} \to 0. \tag{7}$$

Next, using (7) we compute

$$\mathrm{Ext}^1(F_1, \mathcal{O}_Y(-1)) = k.$$

Indeed, F satisfies $H^*(F(-1)) = 0$ and $\mathrm{Ext}^2(\mathcal{O}_Y, \mathcal{O}_L) \simeq H^0(\mathcal{O}_L) = k$. Therefore we get a coherent sheaf F_2, canonically defined by C, fitting into

$$0 \to \mathcal{O}_Y(-1)^{\oplus 2} \to F_2 \to F_1 \to 0, \tag{8}$$

so that (7) and (8) read:

$$0 \to \mathcal{O}_Y(-1)^{\oplus 2} \to F_2 \to F \to \mathcal{O}_L(-1)^{\oplus 2} \to 0.$$

So it remains to check that $F_2 \simeq \sigma_*(\mathcal{Q}^\vee_{\widetilde{Y}})$, which, in view of (6), amounts to write:

$$0 \to F_2 \to \mathcal{O}_Y^{\oplus(d+3)} \to E(1) \to 0.$$

To achieve this, we isolate a canonical extension class:

$$k \subset \mathrm{Ext}^1(E(1), F) \tag{9}$$

To see this, applying $\mathrm{Hom}(-, F(-1))$ to (2) and using the vanishing $H^*(F(-1)) = 0$ we get

$$\mathrm{Ext}^1(E(1), F) \simeq \mathrm{Ext}^2(\mathcal{O}_C(dp), F).$$

Apply now $\mathrm{Hom}(-, \mathcal{O}_C(dp))$ to (3) and note $\mathrm{Ext}^1(\mathcal{O}_C(dp), \mathcal{O}_Y) = 0$. Also, by the local-to-global spectral sequence, we get

$$\mathrm{Ext}^1(\mathcal{O}_C(p) \otimes \mathcal{O}_Y(1), \mathcal{O}_C((d-2)p)) \simeq H^1(\mathcal{H}om(\mathcal{O}_C(dp), \mathcal{O}_C((d-2)p))) = H^1(\mathcal{O}_C(-2p)) = k,$$

and this copy of k fits into $\mathrm{Ext}^2(\mathcal{O}_C(dp), F)$. Then, we get:

$$k = H^1(\mathcal{O}_C(-2p)) \subset \mathrm{Ext}^2(\mathcal{O}_C(dp), F) \simeq \mathrm{Ext}^1(E(1), F).$$

Now, from (9) we construct a canonical extension

$$0 \to F \to \widetilde{F} \to E(1) \to 0. \tag{10}$$

We restricting it to L and show that $E|_L \simeq \mathcal{O}_Z \oplus \mathcal{O}_L(-1)^{\oplus 2}$ by applying the argument used to compute $F|_L$. Then, we see that the torsion-free part of $\widetilde{F}|_L$ is $\mathcal{O}_L(-1)^{\oplus 2} \oplus \mathcal{O}_L^{\oplus(d-1)}$, hence we get a canonical surjection $\widetilde{F} \to \mathcal{O}_L(-1)^{\oplus 2}$, whose kernel we call $\widetilde{F}_1$. This fits into

$$0 \to F_1 \to \widetilde{F}_1 \to E(1) \to 0.$$

We have the vanishing $H^*(E) = 0$, hence $\mathrm{Ext}^*(E(1), \mathcal{O}_Y(-1)) = 0$ so from the previous sequence:

$$\mathrm{Ext}^1(\widetilde{F}_1, \mathcal{O}_Y(-1)) \simeq \mathrm{Ext}^1(F_1, \mathcal{O}_Y(-1)) = k^2.$$

We get a sheaf $\widetilde{F}_2$ fitting as a natural extension:

$$0 \to \mathcal{O}_Y(-1)^{\oplus 2} \to \widetilde{F}_2 \to \widetilde{F}_1 \to 0,$$

This is summarized in the diagram

$$\begin{array}{ccccc}
\mathcal{O}_Y(-1)^{\oplus 2} & \longrightarrow & F_2 & \longrightarrow & F_1 \\
\| & & \downarrow & & \downarrow \\
\mathcal{O}_Y(-1)^{\oplus 2} & \longrightarrow & \widetilde{F}_2 & \longrightarrow & \widetilde{F}_1 \\
 & & \downarrow & & \downarrow \\
 & & E(1) & = & E(1)
\end{array}$$

To finish, we have to prove that $\widetilde{F}_2 \simeq \mathcal{O}_Y^{\oplus(d+3)}$. However, we check that the Chern character of $\widetilde{F}_2$ is $d+3$ and, since $H^*(F_2) = 0$ the central column of the above diagram gives $h^0(\widetilde{F}_2) = h^0(E(1)) = d+3$. Therefore $\widetilde{F}_2 \simeq \mathcal{O}_Y^{\oplus(d+3)}$. □

Remark 7.7. Note that we have:

$$\mathrm{Ext}^1(E(1), \mathcal{O}_Y) \simeq H^2(E(-1))^\vee \simeq H^1(\mathcal{O}_C((2-d)p))^\vee \simeq H^0(\mathcal{O}_C((d-4)p)) = k^{d-3}.$$

Therefore, from (10) we get a canonical surjection

$$\mathrm{Hom}(F, \mathcal{O}_Y) \to \mathrm{Ext}^1(E(1), \mathcal{O}_Y) = k^{d-3}. \tag{11}$$

Note that $\mathrm{Hom}(F, \mathcal{O}_Y) \simeq \mathrm{Hom}(F, \mathcal{O}_Y)^\vee$. In addition, since C is isomorphic to $\mathbb{P}^1 = \mathbb{P}(V)$, the space $H^0(F) = H^0(\mathcal{O}_C((d-2)p)) = S^{d-2}V$ is equipped with a canonical duality. Via this duality, the map (11) gives an injection $k^{d-3} \to \mathrm{Hom}(F, \mathcal{O}_Y)$ and an instanton sheaf F_0 fitting into (4).

Proof of Theorem 7.6 for $d = 2$. For $d = 2$, we replace $\sigma_* \mathcal{Q}_{\widetilde{Y}}^\vee$ with $\mathbf{L}_{\mathcal{O}_Y} E(1)[-1]$. This gives a two-term complex, since the evaluation of global section of $E(1)$ appearing in the exact sequence (6) is no longer surjective. Instead, we get

$$0 \to \mathcal{H}^{-1}(\mathbf{L}_{\mathcal{O}_Y} E(1)) \to \mathcal{O}_Y^{\oplus 5} \to E(1) \to \mathcal{H}^0(\mathbf{L}_{\mathcal{O}_Y} E(1)) \to 0.$$

To identify the terms of this sequence, we note that (2) gives rise to the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{I}_C^{\oplus 2} & = & \mathcal{I}_C^{\oplus 2} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & E & \longrightarrow & \mathcal{O}_Y^{\oplus 2} & \longrightarrow & \mathcal{O}_C(p) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_C(-p) & \longrightarrow & \mathcal{O}_C^{\oplus 2} & \longrightarrow & \mathcal{O}_C(p) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{12}$$

We use the leftmost column to evaluate global sections of $E(1)$. Since the base locus of the pencil of hyperplanes through C is $C \cup L$ and $C \cap L$ is a subscheme of length 2, evaluation of global sections gives a long exact sequence:

$$0 \to \mathcal{O}_Y(-1) \to \mathcal{O}_Y^{\oplus 2} \to \mathcal{I}_C(1) \to \mathcal{O}_L(-1) \to 0.$$

Therefore, the leftmost column of Diagram (12) gives $\mathcal{H}^0(\mathbf{L}_{\mathcal{O}_Y}E(1)) \simeq \mathcal{O}_L(-1)^{\oplus 2}$ and:

$$0 \to \mathcal{H}^{-1}(\mathbf{L}_{\mathcal{O}_Y}E(1)) \to \mathcal{O}_Y^{\oplus 5} \to E(1) \to \mathcal{O}_L(-1)^{\oplus 2} \to 0,$$

with

$$0 \to \mathcal{O}_Y(-1)^{\oplus 2} \to \mathcal{H}^{-1}(\mathbf{L}_{\mathcal{O}_Y}E(1)) \to \mathcal{I}_C \to 0. \tag{13}$$

Note that the gluing object $\sigma_*\mathcal{G}$ fits into the triangle

$$\sigma_*\mathbf{T}^2 \to \sigma_*\mathcal{Q}^\vee_{\tilde{Y}} \to \sigma_*\mathcal{G}.$$

We get a long exact sequence by taking the cohomology with respect to the standard heart:

$$0 \to \mathcal{H}^{-1}(\sigma_*\mathcal{G}) \to \mathcal{H}^0(\sigma_*\mathbf{T}^2) \xrightarrow{\beta} \mathcal{H}^0(\sigma_*\mathcal{Q}^\vee_{\tilde{Y}}) \to \mathcal{H}^0(\sigma_*\mathcal{G}) \to \mathcal{H}^1(\sigma_*\mathbf{T}^2) \xrightarrow{\alpha} \mathcal{H}^1(\sigma_*\mathcal{Q}^\vee_{\tilde{Y}}) \to \mathcal{H}^1(\sigma_*\mathcal{G}) \to 0,$$

where $\mathcal{H}^0(\sigma_*\mathcal{Q}^\vee_{\tilde{Y}}) \cong \mathcal{H}^{-1}(\mathbf{L}_{\mathcal{O}_Y}E(1))$ and from the leftmost column we know the sheaf $\mathcal{O}_Y(-1)^{\oplus 2} \cong \mathcal{H}^{-1}(\mathbf{L}_{\mathcal{O}_Y}(I_C \otimes \mathcal{O}_Y(H)))^{\oplus 2}$. Further note that $\mathcal{H}^0(\sigma_*\mathbf{T}^2) \cong \mathcal{H}^{-1}(\mathbf{L}_{\mathcal{O}_Y}(I_C \otimes \mathcal{O}_Y(H)))^{\oplus 2}$. Then $\mathcal{H}^{-1}(\sigma_*\mathcal{G}) \cong \mathrm{Ker}\beta = 0$. Then we have a long exact sequence

$$\begin{aligned} 0 \to \mathcal{O}_Y(-1)^{\oplus 2} \longrightarrow \mathcal{H}^{-1}(\mathbf{L}_{\mathcal{O}_Y}E(1)) \longrightarrow \mathcal{H}^0(\sigma_*(\mathcal{G})) \longrightarrow \\ \longrightarrow \mathcal{O}_L(-1)^{\oplus 2} \xrightarrow{\alpha} \mathcal{O}_L(-1)^{\oplus 2} \longrightarrow \mathcal{H}^1(\sigma_*(\mathcal{G})) \to 0. \end{aligned}$$

It remains to show that the map α is an isomorphism, then $\mathcal{H}^1(\sigma_*\mathcal{G}) = 0$ and $\sigma_*\mathcal{G}$ is a sheaf and it is isomorphic to $\mathcal{H}^0(\sigma_*\mathcal{G}) \cong I_C$, where C is the line on the quartic double solid Y we start with. To achieve this, by definition, we know $\mathcal{H}^1(\sigma_*\mathbf{T}^2) \cong \mathcal{H}^0(\mathbf{L}_{\mathcal{O}_Y}(I_C \otimes \mathcal{O}_Y(H)))^{\oplus 2}$ and $\mathcal{H}^1(\sigma_*\mathcal{Q}^\vee_{\tilde{Y}}) \cong \mathcal{H}^0(\mathbf{L}_{\mathcal{O}_Y}E(1)) \cong \mathcal{H}^0(\mathbf{L}_{\mathcal{O}_Y}(I_C \otimes \mathcal{O}_Y(H)))^{\oplus 2}$ by the short exact sequence from the leftmost column. Thus α is an isomorphism.

7.2. **Comparing with smooth case.** In [JLZ22], the authors show the gluing object $i^!\mathcal{Q}^\vee_X$ for a smooth prime Fano threefold of genus 8 corresponds to a rank two instanton bundle E on a cubic threefold Y via the Kuznetsov type equivalence $\mathcal{K}u(X) \simeq \mathcal{K}u(Y)$(Conjecture 1.1). Together with the results in this section, they provide evidence for the following conjecture.

Conjecture 7.8. *Let Y_d be a smooth del Pezzo threefold of Picard rank one and degree $d \geq 3$. Then all smooth prime Fano threefold X of index one and genus $g = 2d+2$ such that $\Phi : \mathcal{K}u(X) \simeq \mathcal{K}u(Y)$ is a Kuznetsov-type equivalence is parametrized by rank two instanton bundle E_{d-1} of charge $d-1$ on Y_d and*

$$\Phi(i^!\mathcal{Q}^\vee_X) \cong \mathbf{L}_{\mathcal{O}_Y}E_{d-1},$$

□

7.3. **Existence of family of gluing objects.** Let X be an 1-nodal maximally non-factorial prime Fano threefolds of genus $g = 2d+2, d \geq 2$ constructed from *bridge* in [KS23, Proposition 2.6]. Then by [KS23, Theorem 3.6], there is a family $\mathcal{X} \xrightarrow{f} B$ such that the central fiber $\mathcal{X}_o \cong X$ and the other fiber $\mathcal{X}_b, b \neq o$ is smooth prime Fano threefold of the same genus and their intermediate Jacobians are isomorphic. Furthermore, there is a family of vector bundles $\mathcal{U}_\mathcal{X}$ such that its restriction on central fiber $\mathcal{U}_{\mathcal{X}_o} \cong \mathcal{U}_X$ and $\mathcal{U}_{\mathcal{X}_b}$ is the Mukai bundle for smooth Fano threefold $\mathcal{X}_b$. Moreover, there is a semi-orthogonal decomposition

$$D^b(\mathcal{X}) = \langle \iota_* P_X, \widetilde{\mathcal{A}_\mathcal{X}}, f^*D^b(B), f^*D^b(B) \otimes \mathcal{U}^\vee_\mathcal{X} \rangle,$$

where $\iota : X \hookrightarrow \mathcal{X}$ is the embedding of the central fiber and P_X is the $\mathbb{P}^\infty$-object. The category $\widetilde{\mathcal{A}_\mathcal{X}}$ is the family of categories such that $\mathcal{A}_{\mathcal{X}_o} \simeq \mathcal{K}u(Y_d)$, where Y_d is the del Pezzo threefold of degree d and $\mathcal{A}_{\mathcal{X}_\lfloor}$ is the semi-orthogonal component of $\langle \mathcal{O}_{X_b}, \mathcal{U}^\vee_{X_b} \rangle$ for each smooth prime Fano threefold

X_b of genus $2d+2$ and they are all equivalent to $\mathcal{A}_{\mathcal{X}_o} \simeq \mathcal{K}u(Y_d)$. Note that we can rewrite the semi-orthogonal decomposition of $D^b(\mathcal{X})$ as

$$D^b(\mathcal{X}) = \langle \iota_* P_X, \widetilde{\mathcal{A}_{\mathcal{X}}}, f^*D^b(B) \otimes \mathcal{Q}_{\mathcal{X}}^\vee, f^*D^b(B) \rangle,$$

where $\mathcal{Q}_{\mathcal{X}}^\vee$ is the family of objects such that its restriction to the central fiber is isomorphic to $\mathbf{L}_{\mathcal{O}_X}\mathcal{U}_X^\vee[-1]$ and its restriction to other fiber is the dual of tautological quotient bundle. Define the family of Kuznetsov components $\widetilde{\mathcal{K}u(\mathcal{X})} := \langle \iota_* P_X, \widetilde{\mathcal{A}_{\mathcal{X}}} \rangle$ and denote by $\mathcal{I} : \widetilde{\mathcal{K}u(\mathcal{X})} \hookrightarrow \mathcal{O}_{\mathcal{X}}^\perp$ the inclusion functor. Define a family of objects $\mathbb{E} := \mathbf{R}_{\iota_* P_X} I^! \mathcal{Q}_{\mathcal{X}}^\vee$ over B. It is easy to see that $\mathbb{E}_o \cong \mathbf{R}_{\mathcal{P}_X} i^! \mathcal{Q}_X^\vee$ and $\mathbb{E}_b \cong i^! \mathcal{Q}_{X_b}^\vee$ for $b \neq o$. We prove the following property.

Proposition 7.9. *Let X be an 1-nodal maximally non-factorial prime Fano threefolds of genus $g = 2d+2, d \geq 2$ constructed via* bridge*. Let $\mathbb{E}$ be the family of objects constructed above. Then it induces a family $\mathbb{F}$ such that*

(1) $\mathbb{F}_o \cong I_C$ and $\mathbb{F}_b \cong i^! \mathcal{Q}_{X_b}$ for all $b \neq o$, if $d = 2$;
(2) $\mathbb{F}_o \cong F_0$ and $\mathbb{F}_b \cong E_b$, where F_0 is a rank two stable instanton sheaf on the cubic threefold Y, which is associated with the smooth conic C and E_b is a rank two instanton bundle on Y, if $d = 3$.
(3) $\mathbb{F}_o \cong \mathbf{L}_{\mathcal{O}_Y} F_0$, which is an acyclic extension of non-locally free rank two instanton sheaf F_0, which is associated with the smooth rational curve C of degree $d-1$. And $\mathbb{F}_b \cong \mathbf{L}_{\mathcal{O}_Y} E_{d-1}$, where E_{d-1} is a rank two instanton bundle of charge $d-1$ on Y_d, if $d \geq 4$.

Before proving the proposition, we first show the gluing objects constructed in two ways coincide.

Lemma 7.10. *There is an isomorphism of two objects:*

$$\sigma_* \circ \mathbf{L_T} \circ \mathbf{R_T} \circ \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} i^! \mathcal{Q}_{\widetilde{Y}}^\vee \cong \sigma_* \circ \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} \circ \mathbf{L_T} i^! \mathcal{Q}_{\widetilde{Y}}^\vee.$$

Proof. By Definition 7.5, it remains to show that the object $\sigma_* \circ \mathbf{L_T} \circ \mathbf{R_T} \circ \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} i^! \mathcal{Q}_{\widetilde{Y}}^\vee \cong \sigma_* \mathcal{G}$. We compute the object $\mathbf{R_T} \circ \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} i^! \mathcal{Q}_{\widetilde{Y}}^\vee$. First we apply $\mathbf{R_T} \circ \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)}$ to the triangle $\mathcal{U}_{\widetilde{Y}}[1] \to i^! \mathcal{Q}_{\widetilde{Y}}^\vee \to \mathcal{Q}_{\widetilde{Y}}^\vee$, we get

$$\mathbf{R_T} \circ \mathbf{R}_{\mathcal{O}(E-H)} \mathcal{U}_{\widetilde{Y}}[1] \to \mathbf{R_T} \circ \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} i^! \mathcal{Q}_{\widetilde{Y}}^\vee \to \mathcal{Q}_{\widetilde{Y}}^\vee,$$

by Lemma 7.1(1) and (2).

Note that by the proof of Lemma 7.1, $\mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} \mathcal{U}_{\widetilde{Y}}[1] \cong \mathcal{O}_E(E-F)$. Then, finally we get the exact triangle

$$\mathbf{R_T}\, \mathcal{O}_E(E-F) \to \mathbf{R_T} \circ \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} i^! \mathcal{Q}_{\widetilde{Y}}^\vee \to \mathcal{Q}_{\widetilde{Y}}^\vee,$$

with

$$\mathbf{R_T}\, \mathcal{O}_E(E-F) \to \mathcal{O}_E(E-F) \to \mathbf{T}^{\oplus 2}[2].$$

Then we apply $\mathbf{L_T}$ to the exact triangles above, we get

$$\mathbf{L_T} \circ \mathbf{R_T}\, \mathcal{O}_E(E-F) \to \mathbf{L_T} \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} i^! \mathcal{Q}_{\widetilde{Y}}^\vee \to \mathbf{L_T} \mathcal{Q}_{\widetilde{Y}}^\vee,$$

where $\mathbf{L_T} \circ \mathbf{R_T}\, \mathcal{O}_E(E-F) \cong \mathcal{O}_E(E-F)$, by Lemma 7.3. Then we have the triangle

$$\mathcal{O}_E(E-F) \to \mathbf{L_T} \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} i^! \mathcal{Q}_{\widetilde{Y}}^\vee \to \mathbf{L_T} \mathcal{Q}_{\widetilde{Y}}^\vee.$$

Then we apply σ_* to the triangle above, and we get

$$\sigma_* \mathbf{L_T} \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} i^! \mathcal{Q}_{\widetilde{Y}}^\vee \cong \sigma_* \mathbf{L_T} \mathcal{Q}_{\widetilde{Y}}^\vee,$$

since $\sigma_*\, \mathcal{O}_E(E-F) = 0$ by Proposition 7.4(2). Note that there is an exact triangle

$$\mathbf{T}^2 \to \mathcal{Q}_{\widetilde{Y}}^\vee \to \mathbf{L_T} \mathcal{Q}_{\widetilde{Y}}^\vee,$$

by Lemma 7.3. Thus by applying σ_* to this triangle, we get

$$\sigma_* \mathcal{G} \cong \sigma_* \circ \mathbf{L_T} \circ \mathbf{R_T} \circ \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} i^! \mathcal{Q}^\vee_{\widetilde{Y}}.$$

□

Next, we prove Proposition 7.9.

Proof. First, we have $\pi^* i^! \mathcal{Q}^\vee_X \cong i^! \mathcal{Q}^\vee_{\widetilde{Y}}$, where $\pi : \widetilde{Y} \to X$ is a small resolution. Indeed, the object $i^! \mathcal{Q}^\vee_X$ fits into the exact triangle

$$\mathcal{U}_X[1] \to i^! \mathcal{Q}^\vee_X \to \mathcal{Q}^\vee_X.$$

Apply π^* to the triangle, and we get the triangle

$$\mathcal{U}_{\widetilde{Y}}[1] \to \pi^* i^! \mathcal{Q}^\vee_X \to \mathcal{Q}^\vee_{\widetilde{Y}}.$$

Note that $\mathrm{Ext}^1(\mathcal{Q}^\vee_{\widetilde{Y}}, \mathcal{U}_{\widetilde{Y}}[1]) \cong \mathrm{Ext}^1(\mathcal{U}^\vee_{\widetilde{Y}}, \mathcal{Q}^\vee_{\widetilde{Y}}) = k$, thus the desired result holds. By Theorem 2.7,

$$\pi_* \mathbf{R_T} \circ \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} i^! \mathcal{Q}^\vee_{\widetilde{Y}} \cong \mathbf{R}_{\mathcal{P}_X} i^! \mathcal{Q}^\vee_X.$$

As the pair of functors (π_*, π^*) induces the equivalence of categories: $\mathbf{R_T}\mathcal{B}_{\widetilde{Y}} \simeq \widetilde{\mathcal{A}_X}$, so for an object $F \in \mathbf{R_T}\mathcal{B}_{\widetilde{Y}}$, $\pi^* \circ \pi_*(F) \cong F$. Then we have

$$\pi^* \circ \pi_* \mathbf{R_T} \circ \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} i^! \mathcal{Q}^\vee_{\widetilde{Y}} \cong \mathbf{R_T} \circ \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} i^! \mathcal{Q}^\vee_{\widetilde{Y}} \cong \pi^* \mathbf{R}_{\mathcal{P}_X} i^! \mathcal{Q}^\vee_X.$$

Then we apply $\mathbf{L_T}$ to the above isomorphism, we get

$$\mathbf{L_T} \circ \mathbf{R_T} \circ \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} i^! \mathcal{Q}^\vee_{\widetilde{Y}} \cong \mathbf{L_T} \circ \pi^* \mathbf{R}_{\mathcal{P}_X} i^! \mathcal{Q}^\vee_X \in \mathcal{B}_{\widetilde{Y}}.$$

Finally, we apply σ_* to get

$$\sigma_* \circ \mathbf{L_T} \circ \pi^* \mathbf{R}_{\mathcal{P}_X} i^! \mathcal{Q}^\vee_X \cong \sigma_* \circ \mathbf{L_T} \circ \mathbf{R_T} \circ \mathbf{R}_{\mathcal{O}_{\widetilde{Y}}(E-H)} i^! \mathcal{Q}^\vee_{\widetilde{Y}} \cong \sigma_* \mathcal{G} \in \mathcal{K}u(Y).$$

(1) If $d = 2$, $\mathbb{E}_o \cong \mathbf{R}_{\mathcal{P}_X} i^! \mathcal{Q}^\vee_X$, applying the equivalence $\sigma_* \circ \mathbf{L_T} \circ \pi^* : \widetilde{\mathcal{A}_X} \simeq \mathcal{K}u(Y)$, we get $\sigma_* \mathcal{G}$, which is the ideal sheaf I_C of the conic C on the quartic double solid Y_2, by Theorem 7.6. On the other hand, $\mathbb{E}_b(b \neq o) \cong i^! \mathcal{Q}^\vee_{X_b}$ is the gluing object for smooth Gushel-Mukai threefold X_b.
(2) If $d = 3$. A similar process gives a non-locally free instanton sheaf of rank two on the cubic threefold Y_3, corresponding to $\mathbb{E}_o$. For $b \neq o$, $\mathbb{E}_b$ is the gluing object $i^! \mathcal{Q}^\vee_X$, where X are smooth prime Fano threefolds of index one and degree 14. Applying the Kuznetsov-type equivalence $\Psi : \mathcal{A}_X \simeq \mathcal{K}u(Y_3)$, we get rank two instanton bundles on the cubic threefold Y_3, by [JLZ22, Theorem 8.13].
(3) If $d = 4$ or 5, similar process gives the desired results, by Theorem 7.6.

□

7.4. **Refined Categorical Torelli theorem for 1-nodal Fano threefolds.** Let X be a 1-nodal maximally non-factorial prime Fano threefolds of genus $g \geq 6$ (via *bridge construction* from a del Pezzo threefold), whose Kuznetsov component $\mathcal{K}u(X)$ admits a semi-orthogonal decomposition

$$\mathcal{K}u(X) = \langle \mathcal{P}, \widetilde{\mathcal{A}_X} \rangle.$$

In this section, we prove the isomorphism class X is determined by $\widetilde{\mathcal{A}_X}$ and the distinguished object $\mathbf{R}_{\mathcal{P}} i^! \mathcal{Q}^\vee_X \in \widetilde{\mathcal{A}_X}$.

Theorem 7.11. *Let X, X' be 1-nodal Fano threefolds above and $\Phi : \widetilde{\mathcal{A}_X} \simeq \widetilde{\mathcal{A}_{X'}}$ be the equivalence such that $\Phi(\mathbf{R}_{\mathcal{P}} i^! \mathcal{Q}^\vee_X) \cong \mathbf{R}_{\mathcal{P}'} i^! \mathcal{Q}^\vee_{X'}$, then $X \cong X'$.*

Proof. Let $\Psi:\widetilde{\mathcal{A}_X}\simeq\mathcal{K}u(Y)$ and $\Psi':\widetilde{\mathcal{A}_{X'}}\simeq\mathcal{K}u(Y')$ be the equivalences given by [KS23, Proposition 3.3, Remark 3.4]. Since $\Phi:\widetilde{\mathcal{A}_X}\simeq\widetilde{\mathcal{A}_{X'}}$ is an equivalence, then it induces an equivalence $\widetilde{\Phi}:\mathcal{K}u(Y)\simeq\mathcal{K}u(Y')$ such that $\widetilde{\Phi}\circ\Psi\cong\Psi'\circ\Phi$. By assumption, we have

$$(\Psi')^{-1}\circ\widetilde{\Phi}\circ\Psi(\mathbf{R}_{\mathcal{P}}i^!\mathcal{Q}_X^\vee)\cong\mathbf{R}_{\mathcal{P}'}i^!\mathcal{Q}_{X'}^\vee.$$

Then we get

$$\widetilde{\Phi}\circ\Psi(\mathbf{R}_{\mathcal{P}}i^!\mathcal{Q}_X^\vee)\cong\Psi'(\mathbf{R}_{\mathcal{P}'}i^!\mathcal{Q}_{X'}^\vee).$$

By Theorem 7.6 and Lemma 7.10,

$$\Psi(\mathbf{R}_{\mathcal{P}}i^!\mathcal{Q}_X^\vee)\cong\begin{cases} I_L, & g=6\\ E, & g=8\\ \mathbf{L}_{\mathcal{O}_Y}E_{d-1}, & g=10,12.\end{cases}$$

Where E is a rank two stable instanton sheaf on the correspondent cubic threefold Y_3 and E_{d-1} is a rank two stable instanton sheaf of charge $d-1$ on correspondent del Pezzo threefold of degree d. Note that $\widetilde{\Phi}:\mathcal{K}u(Y_d)\simeq\mathcal{K}u(Y_d)$ is an equivalence of Kuznetsov components of del Pezzo threefold $Y_d, d\geq 2$ Then we get

$$\begin{cases}\widetilde{\Phi}(I_L)\cong I_{L'}, & g=6\\ \widetilde{\Phi}(E)\cong E', & g=8\\ \widetilde{\Phi}(\mathbf{L}_{\mathcal{O}_Y}(E_{d-1}))\cong\mathbf{L}_{\mathcal{O}_Y}(E'_{d-1}), & g=10,12.\end{cases}$$

By [FLZ23, Theorem 7.1, Theorem 8.2] and the image of $\widetilde{\Phi}$ shown above, there is a unique isomorphism $f:Y_d\cong Y'_d, d\geq 2$ such that $\widetilde{\Phi}=f_*:\mathcal{K}u(Y_d)\simeq\mathcal{K}u(Y'_d)$. Then by construction of (acyclic extension of) stable and non-locally free instanton sheaves(Section 7.1), we get $X\cong X'$. □

Corollary 7.12. *Let X be a 1-nodal maximally non-factorial prime Fano threefolds of genus $g=2d+2\geq 6$(constructed via the bridge from del Pezzo threefold of degree $d\geq 2$). Then X_g is uniquely determined by the pair (Y_d,F_d), where*

$$F_d=\begin{cases} I_L, & d=2\\ E, & d=3\\ \mathbf{L}_{\mathcal{O}_Y}(E_{d-1}), & d=4,5.\end{cases}$$

8. Fiber of categorical period map via Bridgeland stable objects

The categorical period map was introduced in [JLLZ21a, Remark 10.2] but was only rigorously defined in a recent paper [KS23, Section 1.7]. Using [JLLZ21a, Theorem 10.3] and [BP23, Theorem 1.9], one describes the fiber of categorical period map over the Kuznetsov component of a general ordinary Gushel-Mukai threefold X(the family of all Gushel-Mukai threefolds X' such that $\mathcal{K}u(X')\simeq\mathcal{K}u(X)$), which is disjoint union of two surfaces, called double EPW surface and double dual EPW surface. It is also shown in [JLLZ21a] that these two surfaces are Bridgeland moduli space of stable objects of certain numerical characters in the Kuznetsov component $\mathcal{K}u(X)$. The semi-orthogonal decomposition of X is given by

$$D^b(X)=\langle\mathcal{K}u(X),\mathcal{Q}_X^\vee,\mathcal{O}_X\rangle,$$

where $\mathcal{Q}_X^\vee$ is the dual of tautological quotient bundle and denote by $i:\mathcal{K}u(X)\hookrightarrow\mathcal{O}_X^\perp$ the inclusion functor. As is shown in [JLLZ21a, Theorem 9.2] and [JLZ22, Theorem 1.3] that a smooth Gushel-Mukai threefold X is uniquely determined by its Kuznetsov component $\mathcal{K}u(X)$ and a distinguished object $i^!\mathcal{Q}_X^\vee$, called *gluing object*. Thus the fiber $\mathcal{P}_{\text{cat}}^{-1}([\mathcal{K}u(X)])$ of the *categorical period* *map* $\mathcal{P}_{\text{cat}}$ over its Kuznetsov components $\mathcal{K}u(X)$ is the family of *gluing objects* $i^!\mathcal{Q}^\vee_{X'} \in \mathcal{K}u(X')$ when X' varies but the equivalence class of Kuznetsov components is fixed. Since the object $i^!\mathcal{Q}^\vee_{X'}$ is Bridgeland stable with respect to an appropriate stability condition σ on $\mathcal{K}u(X)$, this makes the fiber as an open subset of the union of the moduli spaces $\bigcup_{\chi(\mathbf{v},\mathbf{v})=\chi([i^!\mathcal{Q}^\vee_X],[i^!\mathcal{Q}^\vee_X])}\mathcal{M}_\sigma(\mathcal{K}u(X),\mathbf{v})$, where $\chi(-,-)\colon \mathcal{N}(X)\times\mathcal{N}(X)\to\mathbb{Z}$ is the Euler pairing on the numerical Grothendieck group $\mathcal{N}(X)$ of the Kuznetsov component $\mathcal{K}u(X)$. In this section, we apply similar idea to compute the fiber of *categorical period map*(defined in [KS23, Section 1.7]) for 1-nodal maximally non-factorial prime Fano threefolds of genus $g \geq 6$[2].

We follow the style of notation in [KS23, Section 1.6, Section 5]. Denote by $\overline{\mathrm{MFM}_{X_g}}$ the moduli stack of Fano-Mukai pairs for genus g prime Fano threefolds and MFM_{X_g} the substack of smooth Fano-Mukai pairs. Denote by $\mathrm{MFC}_{\mathrm{Y_d}}$ the moduli stack of pairs (Y_d, C) of smooth del Pezzo threefold Y_d of degree $d\geq 2$ and degree $d-1$ rational curve $C\subset Y_d$. Denote by $\overline{\mathrm{MFM}_{X_g}}^{(1)}$ the 1-nodal loci of $\overline{\mathrm{MFM}_{X_g}}$ and $\overline{\mathrm{MFM}_{X_g}}^{\leq 1}$ the at most 1-nodal loci. Denote by $\overline{\mathrm{MFM}_{X_{2d+2},Y_d}}^{(1)}$ the del Pezzo component of $\overline{\mathrm{MFM}_{X_g}}^{(1)}$, which is the loci coming from *bridge construction* from del Pezzo threefold. Denote by $\overline{\mathrm{MFM}_{X_g}}^{\leq 1-\mathrm{mnf}} \subset \overline{\mathrm{MFM}_{X_g}}^{\leq 1}$ the loci consisting of smooth component $\overline{\mathrm{MFM}_{X_g}}$ and 1-nodal maximally non-factorial component. We also denote by $\overline{\mathrm{MFM}_{X_{2d+2},Y_d}} := \mathrm{MFM}_{X_{2d+2}}\bigcup\overline{\mathrm{MFM}_{X_{2d+2},Y_d}}^{(1)}$ the open substack of $\overline{\mathrm{MFM}_{X_{2d+2}}}$. Denote by

$$\mathcal{P}_{\mathcal{A}}:\mathrm{MFM}_{X_g}\to\mathrm{MTrCat}$$

the categorical period map associated with the component $\mathcal{A}_{\mathcal{X}}\subset D^b(\mathcal{X})$ with semi-orthogonal decomposition

$$D^b(\mathcal{X}) = \langle \mathcal{A}_{\mathcal{X}}, f^*D^b(S)\otimes\mathcal{Q}^\vee_{\mathcal{X}}, f^*D^b(S)\rangle,$$

where $f:\mathcal{X}\to S$ is a family of smooth prime Fano threefolds of genus $g\geq 6$. Then we denote by

$$\mathcal{P}_{\widetilde{\mathcal{A}}}:\mathcal{S}\to\mathrm{MTrCat}$$

the categorical period map associated with the component $\widetilde{\mathcal{A}_{\mathcal{X}}}$ with semi-orthogonal decomposition of $D^b(\mathcal{X})$ in [KS23, (46)], where $\mathcal{S}$ is a substack of $\overline{\mathrm{MFM}_{X_g}}$ for a family $(f:\mathcal{X}\to S,\mathcal{U}_{\mathcal{X}})$ of Fano-Mukai pairs.

First, we rephrase the "classical" results for smooth prime Fano threefolds.

Proposition 8.1.

(1) *The fiber of categorical period map*

$$\mathcal{P}_{\mathcal{A}}:\mathrm{MFM}_{X_6}\to\mathrm{MTrCat}.$$

over the Kuznetsov component $\mathcal{K}u(X)$ *of a* general *Gushel-Mukai threefold* X *is the disjoint union of two surfaces:* $\widetilde{Y}^{\geq 2}_{A^\perp}\bigcup\widetilde{Y}^{\geq 2}_{A}$*, where* A *is the corresponding Lagrangian subspace for* X*.*

(2) *The fiber of categorical period map*

$$\mathcal{P}_{\mathcal{A}}:\mathrm{MFM}_{X_8}\to\mathrm{MTrCat}.$$

over the Kuznetsov component $\mathcal{K}u(X)$ *of a smooth genus* 8 *index one prime Fano threefold* X *is the locus of rank two instanton bundle over a cubic threefold* Y*.*

Proof.

[2]we believe the results are known to the expert, we write it down for pointing out its connection to Bridgeland moduli spaces

(1) The fiber of categorical period map consists of all Gushel-Mukai threefold X' such that $\mathcal{K}u(X') \simeq \mathcal{K}u(X)$, which is parametrized by the gluing objects $i^!\mathcal{Q}^\vee_{X'}$ when X' varies. As a result the fiber is contained in Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{K}u(X), \mathbf{v}) \bigcup \mathcal{M}_\sigma(\mathcal{K}u(X), \mathbf{w})$, where $\mathbf{v}, \mathbf{w} \in \mathcal{N}(\mathcal{K}u(X))$ such that $\chi(v, v) = \chi(w, w) = \chi([i^!\mathcal{Q}^\vee_{X'}], [i^!\mathcal{Q}^\vee_{X'}]) = -1$, and by [JLLZ21a, Theorem 9.3] or [FGLZ24, Theorem B.8], the fiber of $\mathcal{P}_{\mathcal{A}}$ is contained in $\widetilde{Y}^{\geq 2}_{A^\perp} \bigcup \widetilde{Y}^{\geq 2}_{A}$. On the other hand, each point in the union of the two surfaces determines a Gushel-Mukai threefold, which is either period partner of period dual of X. Then the desired result holds by duality conjecture in [KS22, Theorem 1.6].

(2) By similar argument above, the fiber of categorical period map is contained in $\bigcup_{\chi(\mathbf{v},\mathbf{v})=\chi([i^!\mathcal{Q}^\vee_X],[i^!\mathcal{Q}^\vee_X])} \mathcal{M}_\sigma(\mathcal{K}u(X), \mathbf{v})$ and $\chi([i^!\mathcal{Q}^\vee_X], [i^!\mathcal{Q}^\vee_X]) = -4$. It is not hard to see that $\bigcup_{\chi(\mathbf{v},\mathbf{v})=\chi([i^!\mathcal{Q}^\vee_X],[i^!\mathcal{Q}^\vee_X])} \mathcal{M}_\sigma(\mathcal{K}u(X), \mathbf{v}) \cong \mathcal{M}_\sigma(\mathcal{K}u(Y), 2[I_l])$, where Y is the cubic threefold such that $\Phi : \mathcal{K}u(X) \simeq \mathcal{K}u(Y)$ is the Kuznetsov's type equivalence in Conjecture 1.1 and $[I_l]$ is numerical class of ideal sheaf of a line $l \subset Y$. Indeed, $\Phi(i^!\mathcal{Q}^\vee_{X'}) \in \mathcal{K}u(Y)$ is actually a rank two instanton bundle on Y. Thus the fiber is contained in moduli space $M^{inst}_Y(2, 0, 2)$ of instanton bundle of rank two on Y. On the other hand, each rank two instanton bundle on Y determines an index one prime Fano threefold X of genus 8 such that $\mathcal{K}u(X) \simeq \mathcal{K}u(Y)$ by [Kuz04a, Theorem 2.9]. Then the desired result holds.

□

Next, we describe the fiber of categorical period map $\mathcal{P}_{\widetilde{\mathcal{A}}}$ for 1-nodal prime Fano threefolds, using Theorem 7.11.

Proposition 8.2.

(1) The fiber of categorical period map

$$\mathcal{P}_{\widetilde{\mathcal{A}}} \colon \overline{\mathrm{MFM}_{\mathrm{X_6,Y_2}}} \to \mathrm{MTrCat}$$

- *over $\widetilde{\mathcal{A}_X}$ of a smooth a smooth Gushel-Mukai threefold X (which is nothing but Kuznetsov component $\mathcal{K}u(X)$) is the disjoint union of two surfaces: $\widetilde{Y}^{\geq 2}_{A^\perp} \bigcup \widetilde{Y}^{\geq 2}_{A}$.*
- *over $\widetilde{\mathcal{A}_X}$, where X is a 1-nodal maximally non-factorial Gushel-Mukai threefold via* bridge construction *from a quartic double solid Y is Hilbert scheme $F(Y)$ of lines on Y*

(2) The fiber of categorical period map

$$\mathcal{P}_{\widetilde{\mathcal{A}}} \colon \overline{\mathrm{MFM}_{\mathrm{X_8,Y_3}}} \to \mathrm{MTrCat}$$

over $\widetilde{\mathcal{A}_X}$ where X is either a smooth index one prime Fano threefold of genus 8 or a 1-nodal maximally non-factorial one via bridge construction *from a cubic threefold Y is isomorphic to the complement of the strictly semistable objects in moduli space $\mathcal{M}_\sigma(\mathcal{K}u(Y), 2[I_l])$ of semistable objects, which consists of rank two instanton bundles and rank two stable but non locally free instanton sheaves.*

(3) For $d \geq 4$, the fiber of categorical period map

$$\mathcal{P}_{\widetilde{\mathcal{A}}} \colon \overline{\mathrm{MFM}_{\mathrm{X_{2d+2},Y_d}}}^{(1)} \to \mathrm{MTrCat}$$

over $\widetilde{\mathcal{A}_X}$ with X to be a 1-nodal maximally non-factorial genus $2d+2$ Fano as above is the locus of acyclic extension of a rank two charge $d-1$ non locally free stable instanton sheaves F_0 of the form

$$0 \to F_0 \to \mathcal{O}_Y^{\oplus 2} \to \mathcal{O}_C((d-2)p) \to 0.$$

Proof. (1) By definition, $\mathcal{P}_{\widetilde{\mathcal{A}}}^{-1}(\mathcal{K}u(X)) \cong \{X'|\mathcal{K}u(X') \simeq \mathcal{K}u(X)\} \bigcup \{X''|\mathcal{K}u(Y) \simeq \mathcal{K}u(X)\}$, where X' is a smooth Gushel-Mukai threefold and X'' is a 1-nodal maximally non factorial Gushel-Mukai threefold via *bridge* from a quartic double solid Y. Note that $\mathcal{K}u(X)$ is never equivalent to $\mathcal{K}u(Y)$ by [Zha20] and [BP23]. Then the fiber is the disjoint union of two surfaces: $\widetilde{Y}^{\geq 2}_{A^{\perp}} \bigcup \widetilde{Y}^{\geq 2}_{A}$ by Proposition 8.1. On the other hand, since X is 1-nodal maximally non-factorial Gushel-Mukai threefold as above, then $\widetilde{\mathcal{A}_X} \simeq \mathcal{K}u(Y)$, where Y is the corresponding quartic double solid. Thus the fiber in question is just $\mathcal{P}_{\widetilde{\mathcal{A}}}^{-1}(\mathcal{K}u(Y))$ consists of all 1-nodal maximally non-factorial Gushel-Mukai threefold X' such that $\widetilde{\mathcal{A}_{X'}} \cong \widetilde{\mathcal{A}_X} \simeq \mathcal{K}u(Y)$, which is parametrized by $\Psi(\mathbf{R}_{\mathcal{P}} i^! \mathcal{Q}^{\vee}_{X'}) \in \mathcal{K}u(Y)$, where $\Psi : \widetilde{\mathcal{A}_X} \simeq \mathcal{K}u(Y)$n is the equivalence in [KS23, Proposition 3.3, Remark 3.4], which is exactly ideal sheaf of a line $L \subset Y$(By Theorem 7.6 and Lemma 7.10). Thus the fiber is contained in $\mathcal{M}_{\sigma}(\mathcal{K}u(Y), [I_l]) \cong F(Y)$. On the other direction, each point in $F(Y)$ gives a line on Y, producing the 1-nodal maximally non-factorial Gushel-Mukai threefold X' such that $\widetilde{\mathcal{A}_{X'}} \simeq \mathcal{K}u(Y)$. Then the desired result holds.

(2) As the image of $\mathcal{P}_{\widetilde{\mathcal{A}}}$ of either 1-nodal maximally non-factorial or smooth prime Fano threefold of index one and genus 8 are both Kuznetsov component of some cubic threefold, say Y, the fiber of categorical period map over $\mathcal{K}u(Y)$ is isomorphic to $\{X'|\mathcal{K}u(X') \simeq \mathcal{K}u(Y)\} \bigcup \{X''|\mathcal{K}u(Y') \simeq \mathcal{K}u(Y)\}$, where X' is smooth prime Fano threefold of index one and genus 8 and Y' is the cubic threefold associated to X''. Then by [JLZ22, Theorem 1.3], those X' is parametrised by $i^! \mathcal{Q}^{\vee}_{X'}$ when $\mathcal{K}u(X')$ is fixed and by Theorem 7.11, those X'' is parametrized by $\mathbf{R}_{\mathcal{P}} i^! \mathcal{Q}^{\vee}_{X''}$ when $\widetilde{\mathcal{A}_{X''}}$ is fixed(up to equivalence of categories). Then the fiber over $\mathcal{K}u(Y)$ is isomorphic to

$$\{i^! \mathcal{Q}^{\vee}_{X'}\} \bigcup \{\mathbf{R}_{\mathcal{P}} i^! \mathcal{Q}^{\vee}_{X''}\} \subset \mathcal{M}_{\sigma}(\mathcal{K}u(Y), 2[I_l])$$

, by [JLZ22, theorem 8.13], Theorem 7.6 and Theorem 9.5. Indeed the fiber is contained in the complement of locus of strictly semistable objects in $\mathcal{M}_{\sigma}(\mathcal{K}u(Y), 2[I_l])$. On the other hand, each point in this locus is either an instanton bundle on Y or a non-locally free and stable instanton sheaf. The previous one determines an index one prime Fano threefold of genus 8 X' such that $\widetilde{\mathcal{A}_{X'}} \simeq \mathcal{K}u(X') \simeq \mathcal{K}u(Y)$ by [Kuz04b]; the latter one determines a 1-nodal maximally non-factorial prime Fano threefold X of genus 8 such that $\widetilde{\mathcal{A}_X} \simeq \mathcal{K}u(Y)$, by [KS23, Setup 2.2]. Then, the desired result holds.

(3) The argument is very similar to the above, using Theorem 7.6, we omit the details.

□

Remark 8.3.

(1) In (1), The fiber of $\mathcal{P}_{\widetilde{\mathcal{A}}}$ over one point is irreducible, while reducible over another point. It would be very interesting to understand this phenomenon.

(2) Denote by $\mathcal{P} : \overline{\mathrm{MFM}_{X_g}}^{\leq 1-\mathrm{mnf}} \to \mathcal{A}$(where $\mathcal{A}$ is the moduli space of 5-dimensional p.p.a.v.)the "classical" period map such that $X \mapsto J(X)$. Then the fiber over any point strictly contains the locus of instanton bundles and stable but non-locally free stable instanton sheaves, whose complement is the locus of the second-type 1-nodal maximally non-factorial prime Fano threefolds of genus 8 in No.7 in the table in [CKGS23]. It is reasonable to conjecture that this locus corresponds to the strictly semistable objects in the moduli space $\mathcal{M}_{\sigma}(\mathcal{K}u(Y), 2[I_l])$, and the fiber of $\mathcal{P}_{\widetilde{\mathcal{A}}}$: $\overline{\mathrm{MFM}_{X_g}}^{\leq 1-\mathrm{mnf}} \to \mathrm{MTrCat}$ over $\widetilde{\mathcal{A}_X}$ should be the whole Bridgeland moduli space.

8.1. **Naive extension of categorical period map.** Instead of considering the categorical period map $\mathcal{P}_{\widetilde{A}}$, we could consider categorical period map $\mathcal{P}_{\mathcal{A}}$ associated with associated with the component $\mathcal{A}_{\mathcal{X}} \subset D^b(\mathcal{X})$ with semi-orthogonal decomposition

$$D^b(\mathcal{X}) = \langle \mathcal{A}_{\mathcal{X}}, f^*D^b(S) \otimes \mathcal{Q}_{\mathcal{X}}^{\vee}, f^*D^b(S) \rangle.$$

In this case, $\mathcal{P}_{\mathcal{A}}$ takes value in a not necessarily smooth and proper category. In this setting, consider the categorical period map

$$\mathcal{P}_{\mathcal{A}} \colon \overline{\mathrm{MFM}_{X_g}}^{\leq 1-\mathrm{mnf}} \to \mathrm{MTrCat}.$$

Then our results on birational categorical Torelli Theorem 6.2 can be re-interpreted as follows.

Theorem 8.4.

(1) *The fiber of $\mathcal{P}_{\mathcal{A}}$ over $\mathcal{K}u(X)$ of a smooth ordinary Gushel-Mukai threefold is isomorphic to the disjoint union of two surfaces: $\widetilde{Y}^{\geq 2}_{A^{\perp}} \bigcup \widetilde{Y}^{\geq 2}_{A}$, while the fiber over Kuznetsov component of the 1-nodal maximally non-factorial Gushel-Mukai threefold is contained in Hilbert scheme $F(Y)$ of lines on a quartic double solid Y.*

(2) *The fiber $\mathcal{P}_{\mathcal{A}}$ over $\mathcal{K}u(X)$ of a smooth index one prime Fano threefold of genus 8 is isomorphic to the locus of rank two instanton bundle on a cubic threefold, while the fiber over that of a 1-nodal maximally non-factorial index one prime Fano threefold of genus 8 is contained in the complement of the locus of instanton bundle in the Bridgeland moduli space $\mathcal{M}_{\sigma}(\mathcal{K}u(Y), 2[I_l])$ on a cubic threefold Y.*

In [KP18], the authors proposed the duality conjecture, stating that smooth Gushel-Mukai varieties that are period partner or period dual admitting the equivalent Kuznetsov components, which is proved in their later work [KP19]. Thus it is tempting to make the following conjecture for 1-nodal maximally non-factorial prime Fano threefolds.

Conjecture 8.5. *Let X and X' be 1-nodal maximally non-factorial prime Fano threefold of genus $g \geq 6$ such that $J(X) \cong J(X')$. Then $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$.*

Remark 8.6. The Conjecture 8.5 implies the fiber of categorical period map in Theorem 8.4 over the 1-nodal maximally non-factorial prime Fano threefold is not only contains in the Bridgeland moduli space but also equal to it.

Our results in Section 7 and Section 8 support the following Conjecture.

Conjecture 8.7. *Let*

$$\mathcal{P}_{\widetilde{\mathcal{A}}} \colon \overline{\mathrm{MFM}_{X_{2d+2}}}^{\leq 1\mathrm{mnf}} \to \mathrm{MTrCat},$$

with $d \geq 3$ be the categorical period map. Then for any point $\widetilde{\mathcal{A}_X}$ in its image, the fiber

$$\mathcal{P}^{-1}_{\widetilde{\mathcal{A}}}(\widetilde{\mathcal{A}_X}) \cong \mathcal{M}_{\sigma}(\mathcal{K}u(Y_d), (d-1)[I_l]).$$

Remark 8.8. For $d = 2$, the conjecture above holds over the point $\widetilde{\mathcal{A}_X}$ for 1-nodal maximally non-factorial Gushel-Mukai threefold X. For $d = 4$, it is not hard to show the acyclic extension of the rank two instanton sheaf associated with a smooth twisted cubic is stable in the Kuznetsov component.

9. Appendix: Stability conditions on Kuznetsov components

In this section we introduce the definition of (weak) stability conditions on a general triangulated category and a special triangulated category, known as the Kuznetsov component of del Pezzo threefold of Picard rank one, which is given by [BLMS17, Proposition 5.1, Proposition 6.9]. Then we recall several important properties of stability conditions on the Kuznetsov components and describe several Bridgeland moduli spaces constructed from them.

9.1. Stability conditions on a general triangulated category.

Definition 9.1. A *stability condition* on a triangulated category $\mathcal{T}$ is a pair $\sigma = (\mathcal{A}, Z)$, where $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{T}$ and $Z\colon \Lambda \to \mathbb{C}$ is a group homomorphism such that

(1) the composition $Z \circ v : \mathrm{K}(\mathcal{A}) \cong \mathrm{K}(\mathcal{T}) \to \mathbb{C}$ is a stability function on $\mathcal{A}$, i.e. for any $E \in \mathcal{A}$, we have $\mathrm{Im}Z(v(E)) \geq 0$ and if $\mathrm{Im}Z(v(E)) = 0$, $\mathrm{Re}Z(v(E)) < 0$. From now on, we write $Z(E)$ rather than $Z(v(E))$.

For any object $E \in \mathcal{A}$, we define the slope function $\mu_\sigma(-)$ as

$$\mu_\sigma(E) := \begin{cases} -\frac{\mathrm{Re}Z(E)}{\mathrm{Im}Z(E)}, & \mathrm{Im}Z(E) > 0 \\ +\infty, & \text{else.} \end{cases}$$

An object $0 \neq E \in \mathcal{A}$ is called σ-(semi)stable if for any proper subobject $F \subset E$, we have $\mu_\sigma(F)(\leq)\mu_\sigma(E)$.

(2) Any object $E \in \mathcal{A}$ has a Harder–Narasimhan filtration in terms of σ-semistability defined above.

(3) There exists a quadratic form Q on $\Lambda \otimes \mathbb{R}$ such that $Q|_{\ker Z}$ is negative definite and $Q(E) \geq 0$ for all σ-semistable objects $E \in \mathcal{A}$. This is known as the *support property*.

The *phase* of a σ-semistable object $E \in \mathcal{A}$ is defined as

$$\phi_\sigma(E) := \frac{1}{\pi}\arg(Z(E)) \in (0, 1].$$

For $n \in \mathbb{Z}$, we set $\phi_\sigma(E[n]) := \phi_\sigma(E) + n$.

A *slicing* $\mathcal{P}_\sigma$ of $\mathcal{T}$ with respect to the stability condition σ consists of full additive subcategories $\mathcal{P}_\sigma(\phi) \subset \mathcal{T}$ for each $\phi \in \mathbb{R}$ such that the subcategory $\mathcal{P}_\sigma(\phi)$ contains the zero object and all σ-semistable objects whose phase is ϕ.

For any interval $I \subset \mathbb{R}$, we denote by $\mathcal{P}_\sigma(I)$ the category given by the extension closure of $\{\mathcal{P}_\sigma(\phi)\}_{\phi \in I}$. We will use both notations $\sigma = (\mathcal{A}, Z)$ and $\sigma = (\mathcal{P}_\sigma, Z)$ for a stability condition σ with heart $\mathcal{A} = \mathcal{P}_\sigma((0, 1])$.

Definition 9.2. Let $\mathcal{T}$ be a triangulated category and Φ be an auto-equivalence of $\mathcal{T}$. We say a stability condition σ on $\mathcal{T}$ is Φ*-invariant* for an element $\widetilde{g} \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$. We say σ is *Serre-invariant* if it is $S_\mathcal{T}$-invariant, where $S_\mathcal{T}$ is the Serre functor of $\mathcal{T}$.

9.2. Stability conditions on Kuznetsov components of del Pezzo threefolds.

In [BLMS17], the authors provide a way to construct stability conditions on $\mathcal{K}u(Y)$ of a del Pezzo threefold Y of Picard rank one from weak stability conditions on $D^b(Y)$.

Theorem 9.3 ([BLMS17])**.** *Let Y be a smooth del Pezzo threefold of Picard rank one. Then there exists a family of stability conditions on $\mathcal{K}u(Y)$.*

When Y is a del Pezzo threefold of degree $d \geq 2$, it is proved in [PR21] that stability conditions on $\mathcal{K}u(Y)$ constructed in [BLMS17] are *Serre-invariant*. Furthermore, they all belong to the same $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-orbit.

Theorem 9.4 ([JLLZ21a, FP21]). *Let Y be a del Pezzo threefold of degree $d \geq 2$, then all Serre-invariant stability conditions on $\mathcal{K}u(Y)$ are contained in the same $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-orbit.*

9.3. **Bridgeland moduli space of stable objects of character $d[I_l]$.** Denote by I_l the ideal sheaf of a line $l \subset Y$ on del Pezzo threefold Y, it is clear that $I_l \in \mathcal{K}u(Y)$ and its character $[I_l] = \mathbf{v} = 1 - L$ in numerical Grothendieck group $\mathcal{N}(\mathcal{K}u(Y))$. In this section we recall the description of Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{K}u(Y_d), (d-1)\mathbf{v})$ of semistable objects of class $(d-1)\mathbf{v}$ in Kuznetsov component $\mathcal{K}u(Y_d)$ of del Pezzo threefold of degree $d-1$, for $d \geq 2$.

Theorem 9.5. *[PY21, Theorem 1.1], [LZ21, Theorem 1.5(1)] Let Y be a del Pezzo threefold of degree d and σ be any Serre-invariant stability condition on $\mathcal{K}u(Y)$. Then*

(1) $\mathcal{M}^{ss}_\sigma(\mathcal{K}u(Y), \mathbf{v}) \cong F(Y)$, if $d \geq 2$, where $F(Y)$ is the Fano surface of lines. Every σ-stable object of class $\mathbf{v}$ is of the form $I_l[2k]$ for some $k \in \mathbb{Z}$.

(2) $\mathcal{M}^{ss}_\sigma(\mathcal{K}u(Y), 2\mathbf{v}) \cong M^{inst}_Y(2,0,2)$ if $d = 3$. Moreover, $E \in \mathcal{M}^{ss}_\sigma(\mathcal{K}u(Y), 2\mathbf{v})$ if and only if E is one of following three forms:

- *instanton bundle.*
- *stable instanton sheaf but not locally free and there is a short exact sequence*

$$0 \to E \to \mathcal{O}_Y^{\oplus 2} \to \mathcal{O}_C(1) \to 0,$$

where C is a smooth conic.

- *strictly semistable sheaf and there is a short exact sequence*

$$0 \to I_l \to E \to I_{l'} \to 0,$$

where l, l' are lines on Y.

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