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# New perspectives on categorical Torelli theorems for del Pezzo threefolds

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ABSTRACT

Let $Y_d$ be a del Pezzo threefold of Picard rank one and degree $d \geq 2$. In this paper, we apply two different viewpoints to study $Y_d$ via a particular admissible subcategory of its bounded derived category, called the Kuznetsov component:

(i) Brill–Noether reconstruction. We show that $Y_d$ can be uniquely recovered as a Brill–Noether locus of Bridgeland stable objects in its Kuznetsov component.
(ii) Exact equivalences. We prove that up to composing with an explicit auto-equivalence, any Fourier–Mukai type equivalence of Kuznetsov components of two del Pezzo threefolds of degree $2 \leq d \leq 4$ can be lifted to an equivalence of their bounded derived categories. As a result, we obtain a complete description of the group of Fourier–Mukai type auto-equivalences of the Kuznetsov component of $Y_d$.

In an appendix, we classify instanton sheaves on quartic double solids, generalizing a result of Druel.


RÉSUMÉ

Soit $Y_d$ un threefold de del Pezzo de rang Picard un et de degré $d \geq 2$. Dans cet article, nous appliquons deux points de vue différents pour étudier $Y_d$ via une sous-catégorie admissible particulière de sa catégorie dérivée bornée, appelée la composante de Kuznetsov :

(i) Reconstruction de Brill–Noether. Nous montrons que $Y_d$ peut être récupéré de manière unique comme un lieu de Brill-Noether d'objets stables de Bridgeland dans sa composante de Kuznetsov.
(ii) Equivalences exactes. Nous prouvons qu'à composition avec une auto-équivalence explicite, toute équivalence de type Fourier-Mukai de composantes de Kuznetsov de deux threefold de del Pezzo de degré $2 \leq d \leq 4$ peut être élevée

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à une équivalence de leurs catégories dérivées bornées. Nous obtenons ainsi une description complète du groupe des auto-équivalences de type Fourier–Mukai de la composante de Kuznetsov de $Y_d$.

Dans une annexe, nous classons les faisceaux instanton sur les solides doubles quartiques, en généralisant un résultat de Druel.


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## 1. Introduction

Let $Y$ be a del Pezzo threefold of Picard rank one, which is an index two prime Fano threefold. By [16], it belongs to one of the five families of threefolds classified by their degree $1 \leq d \leq 5$, see Section 2. By a series of papers of Bondal–Orlov and Kuznetsov, the bounded derived category $\mathrm{D}^b(Y)$ of these Fano threefolds admit a semiorthogonal decomposition

$$\mathrm{D}^b(Y) = \langle \mathcal{K}u(Y), \mathcal{O}_Y, \mathcal{O}_Y(1)\rangle = \langle \mathcal{K}u(Y), \mathcal{Q}_Y, \mathcal{O}_Y\rangle,$$

where $\mathcal{Q}_Y \cong \mathbf{L}_{\mathcal{O}_Y} \mathcal{O}_Y(1)[-1]$ is a rank $d+1$ vector bundle for $d \geq 2$. This paper aims to employ two different viewpoints to extract the critical information of $Y$ from its admissible subcategory $\mathcal{K}u(Y)$, called the Kuznetsov component. Before that, we give the following theorem which is crucial in both directions. Recall that the *rotation functor* $\mathbf{O}$ is an auto-equivalence of $\mathcal{K}u(Y)$ sending $E \in \mathcal{K}u(Y)$ to $\mathbf{L}_{\mathcal{O}_Y}(E \otimes \mathcal{O}_Y(1))$. We denote by $i\colon \mathcal{K}u(Y) \hookrightarrow \mathrm{D}^b(Y)$ the inclusion functor with the right and left adjoints $i^!$ and $i^*$, respectively.

In the following, we consider the object $i^!\mathcal{Q}_Y \in \mathcal{K}u(Y)$, which is the *gluing object* of the semiorthogonal decomposition in the sense of [20]. As explained in [20, Section 2.2], the gluing object together with $\mathcal{K}u(Y)$ encode the information of $\mathcal{O}_Y^{\perp}$. We first show that this gluing object $i^!\mathcal{Q}_Y$ is preserved by any exact equivalence between Kuznetsov components of $Y$ and $Y'$, up to some natural auto-equivalences.

**Theorem 1.1** *(Theorem 7.1). Let $Y$ and $Y'$ be del Pezzo threefolds of Picard rank one and degree $2 \leq d \leq 4$, and $\Phi\colon \mathcal{K}u(Y) \xrightarrow{\simeq} \mathcal{K}u(Y')$ be an exact equivalence.*

(i) *If $2 \leq d \leq 3$, there exist a unique pair of integers $m_1, m_2 \in \mathbb{Z}$ with $0 \leq m_1 \leq 3$ when $d = 2$ and $0 \leq m_1 \leq 5$ when $d = 3$, so that*

$$\Phi(i^!\mathcal{Q}_Y) \cong \mathbf{O}^{m_1}(i'^!\mathcal{Q}_{Y'})[m_2].$$

(ii) *If $d = 4$, there exists a unique pair of integers $m_1, m_2$ and a unique auto-equivalence $T_{\mathcal{L}_0} \in \mathrm{Aut}^0(\mathcal{K}u(Y'))$ (see Section 7.3 for definition) so that*

$$\Phi(i^!\mathcal{Q}_Y) \cong \mathbf{O}^{m_1} \circ T_{\mathcal{L}_0}(i'^!\mathcal{Q}_{Y'})[m_2].$$

*Here* $i' : \mathcal{K}u(Y') \hookrightarrow \mathrm{D}^b(Y')$ *is the inclusion functor.*

To prove degree $2 \leq d \leq 3$ cases, we identify the object $i^!\mathcal{Q}_Y$ via the uniqueness property[1] of it. Up to rotations and shifts, we can assume any exact equivalence $\Phi: \mathcal{K}u(Y) \xrightarrow{\simeq} \mathcal{K}u(Y')$ acts trivially on the numerical Grothendieck group. Take a stable object $E$ in $\mathcal{K}u(Y)$ of the same class as $i^!\mathcal{Q}_Y$, then we show $\mathrm{RHom}(i^*\mathcal{O}_p, E)$ is a two-term complex for all points $p \in Y$ if and only if $E \cong i^!\mathcal{Q}_Y$. Combining it with analysis of the moduli space of stable objects in $\mathcal{K}u(Y)$ of class $[i^*\mathcal{O}_p]$ gives Theorem 1.1. For $d = 4$ case, we use the property of *second Raynaud bundles*.

Then we discuss our two perspectives on categorical Torelli theorems.

**I. Brill–Noether reconstruction.** In [2,1], authors apply stability conditions on $\mathcal{K}u(Y)$ for degree $2 \leq d \leq 3$ to show that one can uniquely recover $Y$ as a subscheme of a moduli space of stable objects in $\mathcal{K}u(Y)$. The following theorem shows that we can describe this subscheme explicitly as a Brill–Noether locus. This generalizes the classical picture for degree $d = 4$, as discussed in Section 6.1.

By [38], [13] and [18], there is a unique Serre-invariant stability condition on $\mathcal{K}u(Y)$ up to the action of $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ for $d \geq 2$, see Section 2. Denote by $\mathcal{M}_\sigma(\mathcal{K}u(Y), v)$ the moduli space[2] of stable objects of a numerical class $v \in \mathcal{N}(\mathcal{K}u(Y))$ with respect to a stability condition $\sigma$ on the Kuznetsov component $\mathcal{K}u(Y)$.

**Theorem 1.2** *(Theorem 6.2). Let $Y$ be a del Pezzo threefold of Picard rank one and degree $d \geq 2$, and let $\sigma$ be a Serre-invariant stability condition on $\mathcal{K}u(Y)$. Then $Y$ is isomorphic to the* Brill–Noether locus[3]

$$\mathcal{BN}_Y := \{F \in \mathcal{M}_\sigma(\mathcal{K}u(Y),\ [i^*\mathcal{O}_p]) :\ \exists k \in \mathbb{Z} \text{ such that } \dim_{\mathbb{C}} \mathrm{Hom}(F[k],\ i^!\mathcal{Q}_Y) \geq d+1\},$$

*where $\mathcal{O}_p$ is the skyscraper sheaf supported at a point $p \in Y$.*

By [38], Serre-invariant stability conditions on $\mathcal{K}u(Y)$ for degree $d \geq 2$ are $\mathbf{O}$-invariant as well. Thus combining Theorem 1.2 and 1.1 gives a new proof for *Categorical Torelli Theorem*.

**Corollary 1.3** *(Corollary 7.11). Let $Y$ and $Y'$ be del Pezzo threefolds of Picard rank one and degree $2 \leq d \leq 4$ such that $\mathcal{K}u(Y) \simeq \mathcal{K}u(Y')$, then $Y \cong Y'$.*

**II. Exact equivalences.** The second viewpoint is to combine the categorical techniques developed in [29] with geometric analysis of stable objects in $\mathcal{K}u(Y)$ to show that any Fourier–Mukai type exact equivalence of Kuznetsov components of two del Pezzo threefolds of degree $2 \leq d \leq 4$ can be lifted to an equivalence of their bounded derived categories.

**Theorem 1.4** *(Theorem 8.2). Let $Y$ and $Y'$ be del Pezzo threefolds of Picard rank one and degree $2 \leq d \leq 4$, and let $\Phi: \mathcal{K}u(Y) \to \mathcal{K}u(Y')$ be an exact equivalence of Fourier–Mukai type such that $\Phi(i^!\mathcal{Q}_Y) = i'^!\mathcal{Q}_{Y'}$. Then $\Phi = f_*|_{\mathcal{K}u(Y)}$ for a unique isomorphism $f: Y \to Y'$.*

---

[1] The object $i^!\mathcal{Q}_Y$ can be viewed as a generalization of the classical notion of second Rayanud bundle over a genus two curve, which is unique up to twisting a line bundle over the curve.

[2] Let $\sigma = (Z, \mathcal{A})$, then up to a shift we may assume $\mathrm{Im}[Z(v)] \geq 0$, then we only consider stable objects in the heart $\mathcal{A}$ to define the moduli space $\mathcal{M}_\sigma(\mathcal{K}u(Y), v)$.

[3] Note that for any $F \in \mathcal{M}_\sigma(\mathcal{K}u(Y),\ d\mathbf{v} - \mathbf{w})$, we prove $\mathrm{RHom}(F, i^!\mathcal{Q}_Y) = \mathbb{C}^\delta[k+1] \oplus \mathbb{C}^{d+\delta}[k]$ where $\delta$ is either zero or one. Hence there exists at most one $k \in \mathbb{Z}$ so that $\dim_{\mathbb{C}} \mathrm{Hom}(F[k],\ i^!\mathcal{Q}_Y) \geq d+1$.

Clearly, combining Theorem 1.1 with Theorem 1.4 provides an alternative proof of *Categorical Torelli theorem* for del Pezzo threefold of degree $2 \leq d \leq 4$. Furthermore, we obtain a complete description of the group $\mathrm{Aut}_{\mathrm{FM}}(\mathcal{K}u(Y))$ of exact auto-equivalences of $\mathcal{K}u(Y)$ of Fourier–Mukai type. For a group $G$ and a subset $S \subset G$, we denote by $\langle S \rangle$ the subgroup of $G$ generated by $S$.

**Corollary 1.5** *(Corollary 8.4). If $Y$ is a del Pezzo threefold of Picard rank one and degree $d$. Then we have*

(1) $\mathrm{Aut}_{\mathrm{FM}}(\mathcal{K}u(Y)) = \langle \mathrm{Aut}(Y), \mathbf{O}, [1] \rangle$ *when* $2 \leq d \leq 3$, *and*
(2) $\mathrm{Aut}_{\mathrm{FM}}(\mathcal{K}u(Y)) = \langle \mathrm{Aut}(Y), \mathrm{Aut}^0(\mathcal{K}u(Y)), \mathbf{O}, [1] \rangle$ *when* $d = 4$.

*Here the subgroup* $\mathrm{Aut}^0(\mathcal{K}u(Y))$ *is defined in Section 7.3.*

We may write elements of $\mathrm{Aut}_{\mathrm{FM}}(\mathcal{K}u(Y))$ in a more explicit way, see Corollary 8.4. Note that by [30, Theorem 1.3], any exact equivalence between Kuznetsov components of quartic double solids is of Fourier–Mukai type. The same also holds for del Pezzo threefolds of degree $d = 4$ as $\mathcal{K}u(Y) \simeq \mathrm{D}^b(C)$ for a smooth curve $C$. Thus in these two cases, $\mathrm{Aut}_{\mathrm{FM}}(\mathcal{K}u(Y)) = \mathrm{Aut}(\mathcal{K}u(Y))$.

**Related work.** Here is the list of relevant results for del Pezzo threefolds $Y_d$ of degree $d$:

$d = 2$. In [10] and [1], the categorical Torelli theorem (Corollary 1.3) has been proved for *generic* quartic double solids. It has been proved for non-generic cases in [7] via Hodge theory for K3 categories. In Theorem 1.2, we give an explicit expression for $Y$ as a Brill–Noether locus of stable objects in $\mathcal{K}u(Y_2)$, and so provide a new proof for the categorical Torelli theorem.

$d = 3$. In [5] and [38], the categorical Torelli theorem has been proved for cubic threefolds by reducing it to classical Torelli theorem. In [28], the author computes the group of auto-equivalences of Kuznetsov components of cubic threefolds of Fourier–Mukai type via a completely different method and provides a new proof of categorical Torelli theorem for cubic threefold by constructing a Hodge isometry between cubic threefolds. In [2], the cubic threefold $Y_3$ has been described geometrically as a sub-locus of a moduli space of stable objects in $\mathcal{K}u(Y_3)$. Theorem 1.2 gives a point-wise description of it as a Brill–Noether locus.

$d = 4$. We know $Y_4$ is the intersection of two quadrics in $\mathbb{P}^5$, and by [35], it can be reconstructed as the moduli space $M$ of stable vector bundles of rank two with fixed determinant of an odd degree over the associated genus two curve $C_2$. We have $\mathcal{K}u(Y_4) \simeq \mathrm{D}^b(C_2)$. As discussed in Section 6.1, our categorical Brill–Noether locus in Theorem 1.2 matches with the classical moduli space $M$.

Other than del Pezzo threefolds, various versions of categorical Torelli theorems are also obtained, see [37] for recent development. In particular, in [19] the authors provide a Brill–Noether reconstruction for index one prime Fano threefolds, and as a result, the refined categorical Torelli theorem is proved.

In [11,39,40,32], a classification of rank two instanton sheaves and the corresponding moduli space in the Kuznetsov component have been discussed for del Pezzo threefolds of degree $d \geq 3$. In Appendix A, we discuss degree $d = 2$ case.

**Organization of the article.** In Section 2, we recall the basic definitions and properties of (weak) stability conditions on del Pezzo threefolds of Picard rank one $Y_d$ of degree $d$ and their Kuznetsov components $\mathcal{K}u(Y_d)$. In particular, we introduce Serre-invariant stability conditions on $\mathcal{K}u(Y_d)$ and describe $\mathcal{K}u(Y_d)$ for each $d \geq 2$. In Section 3, we collect results of general wall-crossing for del Pezzo threefolds which will be used in later sections. In Section 4, we describe the moduli space of $\sigma$-stable objects of the same class as twice of ideal sheaves of lines in the Kuznetsov component of a quartic double solid. In Section 5 we classify $\sigma$-stable objects of the same class as three times of the class of ideal sheaves of lines in the Kuznetsov component of a cubic threefold. In Section 6 we prove Theorem 1.1. In Section 7 we provide a Brill–Noether reconstruction for a del Pezzo threefold of Picard rank one $Y_d$ with respect to $\mathcal{K}u(Y_d)$ and its gluing object $i^!\mathcal{Q}_{Y_d}$, proving Theorem 1.2. Then we prove *categorical Torelli theorem* 1.3. In Section 8 we prove Corollary 1.5. In Appendix A we classify semistable sheaves of rank two, $c_1 = 0, c_2 = 2, c_3 = 0$ on quartic double solids.

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## 2. Background: (weak) Bridgeland stability conditions

In this section, we briefly review the notion of (weak) stability condition on $\mathrm{D}^b(Y)$ and $\mathcal{K}u(Y)$ when $Y := Y_d$ is a del Pezzo threefold of Picard rank one and degree $d$. By [16], every del Pezzo threefold of Picard rank one belongs to the following five families, indexed by their degree $d := H^3 \in \{1, 2, 3, 4, 5\}$:

- $Y_5 = \mathbb{P}^6 \cap \mathrm{Gr}(2, 5)$ is a codimension 3 linear section of Grassmannian $\mathrm{Gr}(2, 5)$.
- $Y_4 = Q \cap Q'$ is intersection of two quadric hypersurfaces in $\mathbb{P}^5$.
- $Y_3 \subset \mathbb{P}^4$ is cubic threefold.
- $Y_2$ is a quartic double solid, i.e. a double cover of $\mathbb{P}^3$ with smooth branch divisor $R \in |\mathcal{O}_{\mathbb{P}^3}(4)|$.
- $Y_1$ is a degree 6 hypersurface of weighted projective space $\mathbb{P}(1, 1, 1, 2, 3)$.

### 2.1. Weak stability conditions on $\mathrm{D}^b(Y)$

For any $b \in \mathbb{R}$, consider the full subcategory of complexes

$$\mathrm{Coh}^b(Y) = \left\{E^{-1} \xrightarrow{d} E^0 : \mu_H^+(\ker d) \leq b, \; \mu_H^-(\mathrm{coker} d) > b\right\} \subset \mathrm{D}^b(Y) \tag{1}$$

Then $\mathrm{Coh}^b(Y)$ is the heart of a bounded t-structure on $\mathrm{D}^b(Y)$ by [8, Lemma 6.1]. For any pair $(b, w) \in \mathbb{R}^2$, we define a group homomorphism $Z_{b,w} : K(Y) \to \mathbb{C}$ by

$$Z_{b,w}(E) := -\mathrm{ch}_2(E)H + w\mathrm{ch}_0(E)H^3 + b(H^2\mathrm{ch}_1(E) - bH^3\mathrm{ch}_0(E)) + \mathfrak{i}\Big(H^2\mathrm{ch}_1(E) - bH^3\mathrm{ch}_0(E)\Big). \tag{2}$$

In [27], the author defined an open region $\widetilde{U} \subset \mathbb{R}^2$ as the set of points $(b, w) \in \mathbb{R}^2$ above the curve $w = \frac{1}{2}b^2 - \frac{3}{8d}$ and above tangent lines of the curve $w = \frac{1}{2}b^2$ at $(k, \frac{k^2}{2})$ for all $k \in \mathbb{Z}$.

In Figs. 1 and 2, we plot the $(b, w)$-plane simultaneously with the image of the projection map

$$\Pi\colon \; K(Y) \setminus \{E\colon \mathrm{ch}_0(E) = 0\} \longrightarrow \mathbb{R}^2,$$

$$E \longmapsto \left(\frac{\mathrm{ch}_1(E).H^2}{\mathrm{ch}_0(E)H^3}, \frac{\mathrm{ch}_2(E).H}{\mathrm{ch}_0(E)H^3}\right).$$

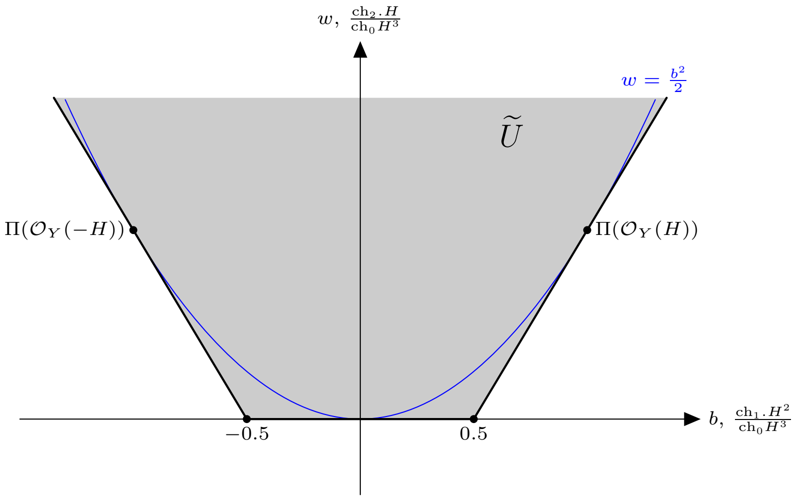


**Fig. 1.** The space $\widetilde{U}$ when $d \leq 3$.

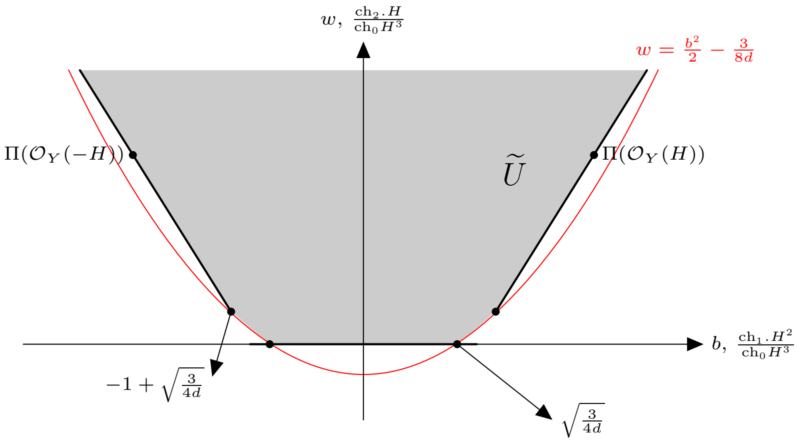


**Fig. 2.** The space $\widetilde{U}$ when $d = 4, 5$.

**Proposition 2.1** *([6, Proposition B.2]). There is a continuous family of weak stability conditions on* $\mathrm{D}^b(Y)$ *parametrized by* $\widetilde{U} \subset \mathbb{R}^2$*, given by*[4]

$$(b, w) \in \widetilde{U} \ \mapsto\ (\mathrm{Coh}^b(Y),\ Z_{b,w}).$$

We now expand upon the above statements. The function $-\frac{\mathrm{Re}[Z_{b,w}(E)]}{\mathrm{Im}[Z_{b,w}(E)]}$ for objects $E \in \mathrm{Coh}^b(Y)$ gives the same ordering as

$$\nu_{b,w}(E) \ = \ \begin{cases} \frac{\mathrm{ch}_2(E).H - w\mathrm{ch}_0(E)H^3}{\mathrm{ch}_1^{bH}(E).H^2} & \text{if } \mathrm{ch}_1^{bH}(E).H^2 \neq 0, \\ +\infty & \text{if } \mathrm{ch}_1^{bH}(E).H^2 = 0, \end{cases} \tag{3}$$

where $\mathrm{ch}^{bH}(E) := \exp(-bH).\mathrm{ch}(E)$.

**Definition 2.2.** Fix a pair $(b, w) \in \widetilde{U}$. We say $E \in \mathrm{D}^b(Y)$ is $\nu_{b,w}$-(semi)stable if and only if

[4] We replaced the pair $(\alpha, \beta)$ with $(w = \frac{1}{2}\alpha^2 + \frac{1}{2}\beta^2, b = \beta)$.

— $E[k] \in \mathrm{Coh}^b(Y)$ for some $k \in \mathbb{Z}$, and
— $\nu_{b,w}(F)\,(\leq)\,\nu_{b,w}\big(E[k]/F\big)$ for all non-trivial subobjects $F \hookrightarrow E[k]$ in $\mathrm{Coh}^b(Y)$.

Here $(\leq)$ denotes $<$ for stability and $\leq$ for semistability.

The image $\Pi(E)$ of $\nu_{b,w}$-semistable objects $E$ with $\mathrm{ch}_0(E) \neq 0$ is *outside* $\widetilde{U}$ by [27, Proposition 3.2], so in particular,

$$\Delta_H(E) \;=\; \big(\mathrm{ch}_1(E).H^2\big)^2 - 2(\mathrm{ch}_0(E)H^3)(\mathrm{ch}_2(E).H) \;\geq\; 0. \tag{4}$$

**Proposition 2.3** *(Wall and chamber structure). Fix $v \in K(Y)$ with $\Delta_H(v) \geq 0$ and $\mathrm{ch}_{\leq 2}(v) \neq 0$. There exists a set of lines $\{\ell_i\}_{i\in I}$ in $\mathbb{R}^2$ such that the segments $\ell_i \cap \widetilde{U}$ (called "*walls of instability*") are locally finite and satisfy*

(a) *If $\mathrm{ch}_0(v) \neq 0$ then all lines $\ell_i$ pass through $\Pi(v)$.*
(b) *If $\mathrm{ch}_0(v) = 0$ then all lines $\ell_i$ are parallel of slope $\frac{\mathrm{ch}_2(v).H}{\mathrm{ch}_1(v).H^2}$.*
(c) *The $\nu_{b,w}$-(semi)stability of any $E \in \mathrm{D}^b(Y)$ of class $v$ is unchanged as $(b, w)$ varies within any connected component (called a "chamber") of $\widetilde{U} \setminus \bigcup_{i\in I} \ell_i$.*
(d) *For any wall $\ell_i \cap \widetilde{U}$, there is an integer $k_i$ and a map $f : F \to E[k_i]$ in $\mathrm{D}^b(Y)$ such that*
  — *for any $(b, w) \in \ell_i \cap \widetilde{U}$, the objects $E[k_i]$, $F$ lie in the heart $\mathrm{Coh}^b(X)$,*
  — *$E$ is $\nu_{b,w}$-semistable of class $v$ with $\nu_{b,w}(E) = \nu_{b,w}(F) = \mathrm{slope}\,(\ell_i)$ constant on the wall $\ell_i \cap \widetilde{U}$, and*
  — *$f$ is an injection $F \hookrightarrow E[k_i]$ in $\mathrm{Coh}^b(Y)$ which strictly destabilizes $E[k_i]$ for $(b, w)$ in one of the two chambers adjacent to the wall $\ell_i$.* □

*2.2. Kuznetsov component*

The Kuznetsov component $\mathcal{K}u(Y)$ is the right orthogonal complement of the exceptional collection $\mathcal{O}_Y, \mathcal{O}_Y(1)$ in $\mathrm{D}^b(Y)$ sitting in the semiorthogonal decomposition

$$\mathrm{D}^b(Y) = \langle \mathcal{K}u(Y), \mathcal{O}_Y, \mathcal{O}_Y(H)\rangle = \langle \mathcal{K}u(Y), \mathcal{Q}_Y, \mathcal{O}_Y\rangle,$$

where $\mathcal{Q}_Y := \mathbf{L}_{\mathcal{O}_Y}\mathcal{O}_Y(1)[-1]$ is a rank $d+1$ vector bundle for $d \geq 2$ (see Section 3.2 for more details). We can identify the numerical Grothendieck group $\mathcal{N}(\mathcal{K}u(Y))$ of $\mathcal{K}u(Y)$ with the image of Chern character map

$$\mathrm{ch}\colon K(\mathcal{K}u(Y)) \to H^*(X, \mathbb{Q}).$$

It is a rank 2 lattice spanned by the classes

$$\mathbf{v} = \left(1,\ 0,\ -\frac{1}{d}H^2,\ 0\right) \qquad \text{and} \qquad \mathbf{w} = \left(0,\ H,\ -\frac{1}{2}H^2,\ \left(\frac{1}{6} - \frac{1}{d}\right)H^3\right).$$

Note that $\mathbf{v}$ is the Chern character of ideal sheaves of lines on $Y$. With respect to this basis, the Euler form on $\mathcal{N}(\mathcal{K}u(Y))$ is represented by the matrix

$$\begin{pmatrix} -1 & -1 \\ 1-d & -d \end{pmatrix}. \tag{5}$$

Consider any admissible subcategory $i\colon \mathcal{C} \hookrightarrow \mathrm{D}^b(Y)$. It has left and right adjoints $i^*$ and $i^!$. Similarly, the embedding $l\colon \mathcal{C}^{\perp} \hookrightarrow \mathrm{D}^b(Y)$ and $r\colon {}^{\perp}\mathcal{C} \hookrightarrow \mathrm{D}^b(Y)$ has left and right adjoints. We know that any object $E \in \mathrm{D}^b(Y)$ lies in the exact triangles

$$r \circ r^!(E) \to E \to i \circ i^*(E) \quad , \quad i \circ i^!(E) \to E \to l \circ l^*(E).$$

We define the right mutation along $\mathcal{C}$ to be the functor

$$\mathbf{R}_{\mathcal{C}} := r \circ r^! \colon \mathrm{D}^b(Y) \to r({}^\perp\mathcal{C})$$

and the left mutation along $\mathcal{C}$ to be

$$\mathbf{L}_{\mathcal{C}} := l \circ l^* \colon \mathrm{D}^b(Y) \to l(\mathcal{C}^\perp).$$

By [22, Proposition 3.8], we know $\mathbf{L}_{\mathcal{C}}|_{r({}^\perp\mathcal{C})}$ and $\mathbf{R}_{\mathcal{C}}|_{l(\mathcal{C}^\perp)}$ are mutually inverse equivalence between the two orthogonal ${}^\perp\mathcal{C} \to \mathcal{C}^\perp$ and $\mathcal{C}^\perp \to {}^\perp\mathcal{C}$. Moreover,

$$(\mathbf{L}_{\mathcal{C}})|_{r({}^\perp\mathcal{C})} = S_{\mathrm{D}^b(Y)} \circ r \circ S^{-1}_{{}^\perp\mathcal{C}} \circ r^* \quad , \quad (\mathbf{R}_{\mathcal{C}})|_{l(\mathcal{C}^\perp)} = S^{-1}_{\mathrm{D}^b(Y)} \circ l \circ S_{\mathcal{C}^\perp} \circ l^*.$$

Here $S_{\mathcal{T}}$ denotes the Serre functor of a triangulated category $\mathcal{T}$ (if it exists).

Let $E \in \mathrm{D}^b(Y)$ be an exceptional object. Then the triangulated subcategory $\langle E \rangle$ generated by $E$ is an admissible subcategory. The embedding functor $i \colon \langle E \rangle \to \mathcal{T}$ has the left and right adjoints

$$i^* = E \otimes \mathrm{RHom}(F, E)^*, \qquad i^!(F) = E \otimes \mathrm{RHom}(E, F).$$

We will abuse notations and write $\mathbf{R}_E$ and $\mathbf{L}_E$ for the corresponding right and left mutations, respectively.

We finish this section by defining the rotation functor. [22, Lemma 4.1, Lemma 4.2] implies that the functor

$$\mathbf{O} \colon \mathrm{D}^b(Y) \to \mathrm{D}^b(Y), \quad \mathbf{O}(-) = \mathbf{L}_{\mathcal{O}_Y}(- \otimes \mathcal{O}_Y(H)) \tag{6}$$

is an auto-equivalence of $\mathcal{K}u(Y)$, called rotation functor. By [38, Remark 5.6], we have

$$S^{-1}_{\mathcal{K}u(Y)} = \mathbf{O}^2[-3].$$

The rotation functor $\mathbf{O}$ induces an auto-isometry of the numerical Grothendieck group $\mathcal{N}(\mathcal{K}u(Y_d))$ for each $d$. In particular for $d = 3$, we have

$$\mathbf{v} \xrightarrow{\mathbf{O}} -2\mathbf{v} + \mathbf{w} \xrightarrow{\mathbf{O}} \mathbf{v} - \mathbf{w} \xrightarrow{\mathbf{O}} \mathbf{v}.$$

And for $d = 2$, we have

$$\mathbf{v} \xrightarrow{\mathbf{O}} -\mathbf{v} + \mathbf{w} \xrightarrow{\mathbf{O}} -\mathbf{v}.$$

### *2.3. Bridgeland stability conditions on $\mathcal{K}u(Y)$*

For any pair $(b, w) \in \widetilde{U}$, consider the tilted heart $\mathrm{Coh}^0_{b,w}(Y) = \langle \mathcal{F}_{b,w}[1], \mathcal{T}_{b,w} \rangle$ where $\mathcal{F}_{b,w}$ ($\mathcal{T}_{b,w}$) is the subcategory of objects in $\mathrm{Coh}^b(X)$ with $\nu^+_{b,w} \leq b$ ($\nu^-_{b,w} > b$). By [3, Proposition 2.14], the pair $\sigma^0_{b,w} := \left(\mathrm{Coh}^0_{b,w}(X), Z^0_{b,w}\right)$ is a weak stability condition on $\mathrm{D}^b(Y)$, where $Z^0_{b,w} := -\mathrm{i}Z_{b,w}$. We denote the corresponding slope function by

$$\mu^0_{b,w}(-) := -\frac{\mathrm{Re}[Z^0_{b,w}(-)]}{\mathrm{Im}[Z^0_{b,w}(-)]}.$$

Let $\mathrm{Coh}_0(X) \subset \mathrm{Coh}(X)$ be the full subcategory consisting of zero-dimensional sheaves.

**Lemma 2.4** *([13, Proposition 4.1]). Any $\sigma^0_{b,w}$-(semi)stable object $E \in \mathrm{Coh}^0_{b,w}(Y)$ is $\nu_{b,w}$-(semi)stable if it does not lie in an exact triangle of the form*

$$F[1] \to E \to T$$

*where $F \in \mathcal{F}_{b,w}$ is $\nu_{b,w}$-(semi)stable and $T \in \mathrm{Coh}_0(X)$. Conversely, take a $\nu_{b,w}$-(semi)stable object $E$ such that either*

(1) $E \in \mathcal{T}_{b,w}$ *and* $\mathrm{Hom}(\mathrm{Coh}_0(X), E) = 0$, *or*
(2) $E \in \mathcal{F}_{b,w}$ *and* $\mathrm{Hom}(\mathrm{Coh}_0(X), E[1]) = 0$.

*Then $E$ is $\sigma^0_{b,w}$-(semi)stable.*

By restricting weak stability conditions $\sigma^0_{b,w}$ to the Kuznetsov component $\mathcal{K}u(Y)$, we obtain stability conditions on it.

**Theorem 2.5** *([3, Theorem 6.8]). For every pair $(b, w)$ in the subset*

$$V := \left\{(b,w) \in \widetilde{U}\colon\ -\frac{1}{2} \leq b < 0,\ w < b^2 \quad or \quad -1 < b < -\frac{1}{2},\ w \leq b^2 + b + \frac{1}{2}\right\} \subset \widetilde{U},$$

*the pair $\sigma(b,w) = (\mathcal{A}(b,w), Z(b,w))$ is a Bridgeland stability condition on $\mathcal{K}u(Y_d)$ where*

$$\mathcal{A}(b,w) := \mathrm{Coh}^0_{b,w}(Y_d) \cap \mathcal{K}u(Y_d) \qquad and \qquad Z(b,w) := Z^0_{b,w}|_{\mathcal{K}u(Y_d)}.$$

**Proof.** Applying the same argument as in the proof of [3, Theorem 6.8] shows that $\sigma(b, w)$ is a Bridgeland stability condition on $\mathcal{K}u(Y_d)$ if $-1 < b < 0$ and

$$\nu_{b,w}(\mathcal{O}_{Y_d}(-2H)[1]) \leq \nu_{b,w}(\mathcal{O}_{Y_d}(-H)[1]) \leq b < \nu_{b,w}(\mathcal{O}_{Y_d}) \leq \nu_{b,w}(\mathcal{O}_{Y_d}(H)). \quad \square$$

On the stability manifold which we denote by $\mathrm{Stab}(\mathcal{K}u(Y))$ we have:

(1) *a right action of the universal covering space* $\widetilde{\mathrm{GL}}^+_2(\mathbb{R})$ *of* $\mathrm{GL}^+_2(\mathbb{R})$: for a stability condition $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(\mathcal{K}u(Y))$ and $\tilde{g} = (g, M) \in \widetilde{\mathrm{GL}}^+_2(\mathbb{R})$, where $g : \mathbb{R} \to \mathbb{R}$ is an increasing function such that $g(\phi+1) = g(\phi)+1$ and $M \in \mathrm{GL}^+_2(\mathbb{R})$, we define $\sigma \cdot \tilde{g}$ to be the stability condition $\sigma' = (\mathcal{P}', Z')$ with $Z' = M^{-1} \circ Z$ and $\mathcal{P}'(\phi) = \mathcal{P}(g(\phi))$ (see [9, Lemma 8.2]).
(2) *a left action of the group of exact auto-equivalences* $\mathrm{Aut}(\mathcal{K}u(Y))$ *of* $\mathcal{K}u(Y)$: for $\Phi \in \mathrm{Aut}(\mathcal{K}u(Y))$ and $\sigma \in \mathrm{Stab}(\mathcal{K}u(Y))$, we define $\Phi \cdot \sigma = (\Phi(\mathcal{P}), Z \circ \Phi_*^{-1})$, where $\Phi_*$ is the automorphism of $K(\mathcal{K}u(Y))$ induced by $\Phi$.

**Remark 2.6.** Note that all stability conditions $\sigma(b, w)$ for $(b, w) \in V$ lie in the same orbit with respect to the action of $\widetilde{\mathrm{GL}}^+_2(\mathbb{R})$ by [38, Proposition 3.5].[5] Hence if $E \in \mathcal{K}u(Y_d)$ is $\sigma(b, w)$-(semi)stable with respect to some $(b, w) \in V$, then it is $\sigma(b, w)$-(semi)stable with respect to any $(b, w) \in V$.

We now give a case-by-case investigation of the category $\mathcal{K}u(Y_d)$ when $d \geq 2$:

[5] This is proved for $V \cap U$, but the same proof is valid for $V$.

$d = 5$. $Y_5$ is a linear section of codimension 3 of $\mathrm{Gr}(2,5)$. Let $\mathcal{U}$ be the restriction of the tautological rank 2 subbundle from $\mathrm{Gr}(2,5)$ to $Y_5$, and let $\mathcal{U}^\perp = \ker(\mathcal{O}_Y \otimes \mathrm{Hom}(\mathcal{O}_Y, \mathcal{U}^*) \to \mathcal{U}^*)$, then [24, Lemma 4.1] gives

$$\mathcal{K}u(Y_5) = \langle \mathcal{U},\, \mathcal{U}^\perp \rangle.$$

$d = 4$. $Y_4$ is an intersection of 2 quadrics in $\mathbb{P}^5$. By [24, Theorem 5.1], there exists a curve $C$ of genus 2 such that we have an equivalence $\mathcal{K}u(Y_4) \cong \mathrm{D}^b(C_2)$. Hence, there is a unique Bridgeland stability condition on $\mathcal{K}u(Y_4)$ up to the action of $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ by [33].

$d = 3$. $Y_3$ is a cubic 3-fold, and $\mathcal{K}u(Y_3)$ is a fractional Calabi–Yau category of dimension $\frac{5}{3}$, i.e. $S^3_{\mathcal{K}u(Y_3)} = [5]$. Note that by [22, Lemma 4.1, Lemma 4.2], we have $S^{-1}_{\mathcal{K}u(Y_3)} = \mathbf{O}^2[-3]$. In this case, we only consider Serre-invariant stability conditions on $\mathcal{K}u(Y_3)$, i.e. those $\sigma \in \mathrm{Stab}(\mathcal{K}u(Y_3))$ so that $S_{\mathcal{K}u(Y_3)}.\sigma = \sigma.\tilde{g}$ for some $\tilde{g} \in \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$. By [38], all stability conditions constructed in Theorem 2.5 are Serre-invariant. And it is proved in [13, Sections 4 & 5] and [18, Theorem 4.25] that all Serre-invariant stability conditions on $\mathcal{K}u(Y_3)$ lie in the same orbit with respect to the action of $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$.

$d = 2$. $Y_2$ is a double cover of $\mathbb{P}^3$ ramified in a quartic surface, called a quartic double solid. By [25, Corollary 4.6], the Serre functor of $\mathcal{K}u(Y_2)$ is $S_{\mathcal{K}u(Y_3)} = \tau[2]$ where $\tau$ is the auto-equivalence of $\mathcal{K}u(Y_2)$ induced by the involution $\tau$ of the double covering. As the involution $\tau$ preserves $\mathrm{Coh}(X)$ and Chern characters, the stability conditions $\sigma(b,w)$ constructed in Theorem 2.5 are Serre-invariant, see [38, Lemma 6.1]. Moreover, [13, Theorem 3.2 & Remark 3.8] and [18, Theorem 4.25] implies that all Serre-invariant stability conditions on $\mathcal{K}u(Y_2)$ lie in the same orbit with respect to action of $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$.

## 3. Del Pezzo threefolds of Picard rank one

In this section, we gather all results which are valid for del Pezzo threefold $Y$ of Picard rank one and degree $d$. By [23], for any $E \in \mathrm{D}^b(Y)$, we know

$$\chi(\mathcal{O}_Y, E) = \mathrm{ch}_0(E) + H^2\mathrm{ch}_1(E)\frac{d+3}{3d} + H\mathrm{ch}_2(E) + \mathrm{ch}_3(E).$$

### *3.1. Instanton bundles and their acyclic extensions*

An instanton of charge $n \geq 2$ on $Y$ is a Gieseker-stable vector bundle $E$ with $\mathrm{ch}_{\leq 2}(E) = (2, 0, -n\frac{H^2}{d})$ satisfying instanton condition $H^1(Y, E(-1)) = 0$. By [24, Lemma 3.5], for each instanton bundle $E$, we have $h^1(E) = n - 2$, thus there exists a unique short exact sequence

$$0 \to E \to \tilde{E} \to \mathcal{O}_Y^{n-2} \to 0$$

such that $\tilde{E}$ is acyclic, i.e. $H^i(Y, \tilde{E}) = 0$ for any $i$. Note that $\tilde{E} = \mathbf{L}_{\mathcal{O}_Y} E$ and is of Chern character

$$n\mathbf{v} = \left(n,\ 0,\ -n\frac{H^2}{d},\ 0\right).$$

Moreover, it is $\nu_{b,w}$-semistable for $b < 0$ and $w \gg 0$.

Let $\ell_d$ be the line passing through $\Pi(n\mathbf{v}) = (0, -\frac{1}{d})$ and $\Pi(\mathcal{O}_Y(-H)) = (-1, \frac{1}{2})$, so it is of equation $w = -\frac{d+2}{2d}b - \frac{1}{d}$. If $d = 2$, then $\ell_d$ coincides with the boundary of $\widetilde{U}$, and if $d \geq 3$, then it intersects $\partial\widetilde{U}$ at two points with $b$-values $b_1^d < b_2^d$ so that

$$b_1^d \leq -1 \qquad \text{and} \qquad -\frac{2}{d+2} = b_2^d. \tag{7}$$

**Lemma 3.1.** *Take a class $\alpha \in K(X)$ with $\mathrm{ch}_{\leq 2}(\alpha) = n\left(1, 0, -\frac{H^2}{d}\right)$ such that $n \leq d+1$. Then there is no wall for class $\alpha$ above $\ell_d$. In particular, an object $E \in \mathrm{Coh}^b(Y)$ of Chern character $\alpha$ which is $\nu_{b,w}$-semistable for $b < 0$ and $w \gg 0$ satisfies $\mathrm{RHom}(\mathcal{O}_Y, E) = \mathrm{Hom}(\mathcal{O}_Y, E[1])[-1]$ and hence $\mathrm{ch}_3(E) \leq 0$.*

**Proof.** Suppose for a contradiction that there is such a wall $\ell$ for class $\alpha$ above $\ell_d$ with the destabilizing sequence $E_1 \to E \to E_2$. Let $b_1 < b_2$ be the intersection points of $\ell$ with the boundary $\partial\widetilde{U}$. Then for $i = 1, 2$,

$$\mu_H^+(\mathcal{H}^{-1}(E_i)) \leq \; b_1 \qquad \text{and} \qquad b_2 \leq \; \mu_H^-(\mathcal{H}^0(E_i)).$$

Let $(r, cH) = \mathrm{ch}_{\leq 1}(\mathcal{H}^{-1}(E_1)) + \mathrm{ch}_{\leq 1}(\mathcal{H}^{-1}(E_2))$, then $(r+n, cH) = \mathrm{ch}_{\leq 1}(\mathcal{H}^0(E_1)) + \mathrm{ch}_{\leq 1}(\mathcal{H}^0(E_2))$, so

$$b_2(r+n) \leq c \; \leq \; b_1 r. \tag{8}$$

Note that if $\mathrm{rk}(\mathcal{H}^{-1}(E_i)) = 0$, then $\mathcal{H}^{-1}(E_i) = 0$. If $d = 2$, then $\ell_d$ lies on the boundary $\partial\widetilde{U}$, so we have $b_1 < -\frac{3}{2}$ and $-\frac{1}{2} < b_2$, so (8) gives $-\frac{1}{2}(r+n) < c < -\frac{3}{2}r$ which has no solution for $n \leq 3$. If $d \geq 3$, then combining (7) and (8) gives $-\frac{2}{d+2}(r+n) < c < -r$ which is not possible for $k \leq d+1$.

For the second claim, we know $E$ is semistable at the large volume limit, so $\mathrm{Hom}(\mathcal{O}_Y, E) = 0$. Also the first part implies that $E$ is $\nu_{b,w}$-semistable for all $(b, w) \in \widetilde{U}$ over $\ell_d$. Since the line segment connecting $\Pi(E)$ and $\Pi(\mathcal{O}_Y(-2))$ is above $\ell_d$, we have $\mathrm{Hom}(E, \mathcal{O}_Y(-2H)[1]) = \mathrm{Hom}(\mathcal{O}_Y, E[2]) = 0$. And we know that $\mathrm{Hom}(\mathcal{O}_Y, E[i]) = \mathrm{Hom}(E, \mathcal{O}_Y(-2)[3-i]) = 0$ for $i \neq 1$. Thus $\chi(E) = -\hom(\mathcal{O}_Y, E[1]) = \mathrm{ch}_3(E) \leq 0$, which gives $\mathrm{ch}_3(E) \leq 0$. □

As a result of the lemma below, we may identify Gieseker-stable sheaves with the large volume limit stable ones.

**Lemma 3.2.** *Let $E$ be an object of class $\mathrm{ch}(E) = n\mathbf{v}$ where $1 \leq n \leq d+2$. Then $E$ is $\nu_{b,w}$-(semi)stable for $b < 0$ and $w \gg 0$ (or equivalently, 2-Gieseker-(semi)stable) if and only if $E$ is a Gieseker-(semi)stable sheaf.*

**Proof.** By [2, Proposition 4.8], the 2-Gieseker-(semi)stability for $E$ coincides with $\nu_{b,w}$-(semi)stability for $b < 0$ and $w \gg 0$. Then in the following we will show 2-Gieseker-(semi)stability for $E$ coincides with Gieseker-(semi)stability

It is clear that if $E$ is 2-Gieseker-stable, then $E$ is Gieseker-stable. Conversely, if $E$ is Gieseker-stable but strictly 2-Gieseker-semistable, then we can find an exact sequence $0 \to E_1 \to E \to E_2 \to 0$ such that $E_i$ are 2-Gieseker-semistable of classes $\mathrm{ch}(E_i) = (k_i, 0, \frac{k_i}{d}H^2, m_i)$, where $1 \leq k_i \leq n-1 \leq d+1$ and $m_i \in \mathbb{Z}_{\leq 0}$. By the stability of $E_i$, we have $m_i \leq 0$ for any $i$ from Lemma 3.1. Since $m_1 + m_2 = 0$, we have $\mathrm{ch}(E_i) = k_i\mathbf{v}$ and contradicts the Gieseker-stability of $E$.

And it is clear that if $E$ is Gieseker-semistable, then $E$ is 2-Gieseker-semistable. Now assume that $E$ is 2-Gieseker-semistable but not Gieseker-semistable. Then the maximal destabilizing subsheaf $E_1$ of $E$ with respect to Gieseker-semistability has class $\mathrm{ch}(E_1) = (k_1, 0, -\frac{k_1}{d}H^2, m_1)$ where $1 \leq k_1 < n$ and $m_i \in \mathbb{Z}_{>0}$. But this contradicts Lemma 3.1 as well. □

### *3.2. The bundle $\mathcal{Q}_Y$ and its projection*

For any smooth Fano threefold $Y$ of index 2 and degree $d \geq 2$, we define the sheaf $\mathcal{Q}_Y$ to be the kernel of the following evaluation map

$$0 \to \mathcal{Q}_Y \to \mathcal{O}_Y \otimes \mathrm{Hom}(\mathcal{O}_Y, \mathcal{O}_Y(1)) \xrightarrow{ev} \mathcal{O}_Y(1) \to 0. \tag{9}$$

We have

$$\mathrm{ch}(\mathcal{Q}_Y) = \left(d+1,\ -H,\ -\frac{1}{2}H^2,\ -\frac{1}{6}H^3\right). \tag{10}$$

**Lemma 3.3.** *The sheaf $\mathcal{Q}_Y$ is a $\mu_H$-stable locally-free sheaf.*

**Proof.** When the degree $d$ of $Y$ satisfies $d \geq 2$, $\mathcal{O}_Y(1)$ has no base-point by [17, Theorem 2.4.5.(i)], hence $\mathcal{Q}_Y$ is a bundle of rank $d+1$. If it is not $\mu_H$-stable, there is a stable reflexive sheaf $Q' \subset \mathcal{Q}_Y$ of bigger or equal slope, thus $\mu_H(Q') \geq 0$. Since it is also a subsheaf of $\mathcal{O}_Y^{\oplus h^0(\mathcal{O}_Y(1))}$ and all stable factors of the latter are the direct sum of $\mathcal{O}_Y$, we get $Q'$ is a direct sum of $\mathcal{O}_Y$ which is not possible as $h^0(\mathcal{Q}_Y) = 0$ by definition. □

Consider the semiorthogonal decomposition $\mathrm{D}^b(Y) = \langle \mathcal{K}u(Y), \mathcal{O}_Y, \mathcal{O}_Y(1)\rangle$. We know $\mathcal{Q}_Y \cong \mathbf{L}_{\mathcal{O}_Y} \mathcal{O}_Y(1)[-1]$. Consider the embedding $i\colon \mathcal{K}u(Y) \hookrightarrow \mathrm{D}^b(Y)$. We know $Q_Y \in \langle \mathcal{O}_Y(-1), \mathcal{K}u(Y)\rangle$, thus it lies in the exact triangle

$$i^!\mathcal{Q}_Y = \mathbf{R}_{\mathcal{O}_Y(-1)}(\mathcal{Q}_Y) \to \mathcal{Q}_Y \to \mathcal{O}_Y(-1) \otimes \mathrm{RHom}(\mathcal{Q}_Y, \mathcal{O}_Y(-1))^\vee.$$

The $\mu_H$-stability of $\mathcal{Q}_Y$ implies that $\mathrm{Hom}(\mathcal{Q}_Y, \mathcal{O}_Y(-1)[k]) = 0$ for $k = 0, 3$. Taking $\mathrm{Hom}(\mathcal{O}_Y(1), -)$ from the exact sequence (9) implies that $\mathrm{hom}(\mathcal{Q}_Y, \mathcal{O}_Y(-1)[1]) = \mathrm{hom}(\mathcal{O}_Y(1), \mathcal{Q}_Y[2]) = 0$. Thus

$$\mathrm{hom}(\mathcal{Q}_Y, \mathcal{O}_Y(-1)[2]) = \chi(\mathcal{Q}_Y, \mathcal{O}_Y(-1)) = 1. \tag{11}$$

Hence $i^!\mathcal{Q}_Y$ is a two-term complex lying in the exact triangle

$$\mathcal{O}_Y(-1)[1] \to i^!\mathcal{Q}_Y \to \mathcal{Q}_Y \tag{12}$$

which is of Chern character $\mathrm{ch}(i^!\mathcal{Q}_Y) = d\mathbf{v}$. In Sections 4 and 5 we show that if $d = 2$ and $d = 3$, the object $i^!\mathcal{Q}_Y$ is Bridgeland-stable in $\mathcal{K}u(Y)$ and it is the only such object which is not Gieseker-stable.

## 4. Moduli spaces on quartic double solids

In this section, we always fix $Y$ to be a del Pezzo threefold of degree two, i.e. a quartic double solid. We aim to classify Bridgeland semistable objects of class $2\mathbf{v}$ in $\mathcal{K}u(Y)$ as described in the following.

**Proposition 4.1.** *Let $\sigma$ be a Serre-invariant stability condition on $\mathcal{K}u(Y)$ and $E \in \mathcal{K}u(Y)$ be a $\sigma$-(semi)stable object of class $2\mathbf{v}$. Then up to a shift, $E$ is either a Gieseker-(semi)stable sheaf or $i^!\mathcal{Q}_Y$.*

**Proof.** By the uniqueness of Serre-invariant stability condition, we can assume that $E \in \mathcal{A}(b, w)$ is a $\sigma(b, w)$-(semi)stable object of class[6] $-2\mathbf{v}$. We divide the proof into several cases. Some lemmas used in this proof will be presented later.

**Step 1.** First we assume that $E$ is $\sigma^0_{b_0,w_0}$-semistable for some $(b_0, w_0) \in V$. Then by Lemma 2.4, we have an exact sequence in $\mathrm{Coh}^0_{b_0,w_0}(Y)$

$$F[1] \to E \to T,$$

[6] We put the shifted class $-2\mathbf{v}$ to get sure $\mathrm{Im}[Z(b, w)] \geq 0$ for $(b, w) \in V$.

where $F \in \mathrm{Coh}^{b_0}(Y)$ with $\nu^+_{b_0,w_0}(F) \leq b$ and $T = 0$ or supported on points. Now by the $\sigma^0_{b_0,w_0}$-semistability of $E$, we know that $F$ is $\nu_{b_0,w_0}$-semistable. By Lemma 3.1, $F$ is $\nu_{b_0,w}$-semistable for $w \gg 0$ and $\mathrm{ch}_3(F) \leq 0$, which implies $T = 0$ and $F[1] = E$. Thus $E[-1]$ is $\nu_{b,w}$-semistable for $w \gg 0$, which implies that $E[-1]$ is a Gieseker-semistable sheaf by Lemma 3.2.

**Step 2.** Now we assume that $E$ is not $\sigma^0_{b,w}$-semistable for any $(b, w) \in V$. By Proposition 2.3, we can assume that there is an open ball $U' \subset \mathbb{R}^2$ containing the point $(b, w) = (-1, \frac{1}{2})$ such that for any $(b, w) \in U_{-1,\frac{1}{2}} := U' \cap V$, we have $E \in \mathcal{A}(b, w)$ and the Harder–Narasimhan filtration of $E$ with respect to $\sigma^0_{b,w}$ is constant.

Let $B$ be the destabilizing quotient object of $E$ with minimum slope and $A \to E \to B$ be the destabilizing sequence of $E$ with respect to $\sigma^0_{b,w}$ for $(b, w) \in U_{-1,\frac{1}{2}}$. Hence $A, B \in \mathrm{Coh}^0_{b,w}(Y)$, which gives

$$\mathrm{Im}(Z^0_{b,w}(E)) \geq \mathrm{Im}(Z^0_{b,w}(B)) > 0, \quad \mathrm{Im}(Z^0_{b,w}(E)) > \mathrm{Im}(Z^0_{b,w}(A)) \geq 0 \tag{13}$$

for all $(b, w) \in U_{-1,\frac{1}{2}}$. Since $\mathrm{Im}(Z^0_{-1,\frac{1}{2}}(E)) = 0$, by the continuity, we have $\mathrm{Im}(Z^0_{-1,\frac{1}{2}}(A)) = \mathrm{Im}(Z^0_{-1,\frac{1}{2}}(B)) = 0$. Therefore, if we assume that $\mathrm{ch}_{\leq 2}(B) = (x, yH, \frac{z}{2}H^2)$ for $x, y, z \in \mathbb{Z}$, from $\mathrm{Im}(Z^0_{-1,\frac{1}{2}}(B)) = 0$ we get $z = -x - 2y$. Thus we have

$$\mathrm{ch}_{\leq 2}(B) = \left(x,\ yH,\ \frac{-x-2y}{2}H^2\right), \quad \mathrm{ch}_{\leq 2}(A) = \left(-2-x,\ -yH,\ \frac{x+2y+2}{2}H^2\right) \tag{14}$$

and by (13) we get

$$1 - 2b^2 + 2w = \mathrm{Im}(Z^0_{b,w}(E)) \ \geq\ \mathrm{Im}(Z^0_{b,w}(B)) = (2b^2 - 2w - 1)\frac{x}{2} - (b+1)y > 0 \tag{15}$$

for all $(b, w) \in U_{-1,\frac{1}{2}}$. Moreover, by definition we have $\mu^0_{b,w}(E) > \mu^0_{b,w}(B)$ for any $(b, w) \in U_{-1,\frac{1}{2}}$ where $\mu^0_{b,w}(-) = -\frac{\mathrm{Re}[Z^0_{b,w}(-)]}{\mathrm{Im}[Z^0_{b,w}(-)]}$, thus

$$\frac{-2b}{1 - 2b^2 + 2w} = \mu^0_{b,w}(E) \ >\ \mu^0_{b,w}(B) = \frac{(bx - y)}{(2b^2 - 2w - 1)\frac{x}{2} - (b+1)y}. \tag{16}$$

Now by (15), $b < 0$ and (16), we have

$$-2b > bx - y. \tag{17}$$

On the other hand, from [3, Remark 5.12], we have

$$\left(\mu^0_{b,w}\right)^-(E) := \mu^0_{b,w}(B) \geq \min\{\mu^0_{b,w}(E),\ \mu^0_{b,w}(\mathcal{O}_Y),\ \mu^0_{b,w}(\mathcal{O}_Y(1))\}$$

for any $(b, w) \in V$. Note that $\mu^0_{-1,\frac{1}{2}}(\mathcal{O}_Y) = -2$, $\mu^0_{-1,\frac{1}{2}}(\mathcal{O}_Y(1)) = -1$ and $\mu^0_{b,w}(E) > 0$ when $(b, w) \in U_{-1,\frac{1}{2}}$ as $\mathrm{Re}[Z^0_{b,w}(E)] = 2b < 0$, thus $\mu^0_{b,w}(B) \geq -2$. By taking the limit $b \to -1$ and $w \to \frac{1}{2}$ and combining with (17), we get

$$2 \geq -x - y \geq 0.$$

**Case 1.** $-x - y = 0$. Then (16) for $-y = x$ gives

$$\frac{-2b}{1 - 2b^2 + 2w} > \frac{(b+1)}{(2b^2 - 2w - 1)\frac{1}{2} + (b+1)},$$

which has no solution for $(b,w)\in V$.

**Case 2.** $-x-y=1$. Then $\mathrm{ch}_{\leq 2}(B)=(x,(-x-1)H,(\frac{x}{2}+1)H^2)$. Since $B$ is $\sigma^0_{b,w}$-semistable, Lemma 2.4 implies that $\mathrm{ch}_{\leq 2}(B)$ is a possible class for $\mathrm{ch}_{\leq 2}$ of a $\nu_{b,w}$-semistable object $B'[1]$ where $B'\in \mathrm{Coh}^b(Y)$. By [27, Proposition 3.2], the only possible cases are $x=\pm 1$ and $\pm 2$. Using (16), we get $x=-2$ and other cases are ruled out. Then we see $\mathrm{ch}_{\leq 2}(B')=(-2,H,0)$. But then $\nu_{b,w}$-semistability of $B'$ for $(b,w)\in U_{-1,\frac{1}{2}}$ and wall and chamber structure described in Proposition 2.3 implies that $B'$ is $\nu_{b=-1,w}$-semistable when $\frac{1}{2}<w<\frac{1}{2}+\epsilon$. Since there is no wall for $B'$ crossing the vertical line $b=-1$, we get $B'$ is $\nu_{b=-1,w}$-semistable for $w\gg 0$. Thus $B'$ is a $\mu_H$-stable sheaf which is not possible by the following Lemma 4.2.

**Case 3.** $-x-y=2$. Then we have $\mathrm{ch}_{\leq 2}(B)=(x,(-x-2)H,(\frac{x}{2}+2)H^2)$. By [27, Proposition 3.2], we have $|x|\leq 3$. Using (16), we get $x=-3$ and other cases are ruled out. Then $\mathrm{ch}_{\leq 2}(B)=(-3,H,\frac{1}{2}H^2)$. We claim that $\mathrm{RHom}(\mathcal{O}_Y,B)=0$, which implies $\mathrm{ch}(B)=(-3,H,\frac{1}{2}H^2,\frac{1}{6}H^3)$. Indeed, since $\mathcal{O}_Y,\mathcal{O}_Y(-2)[2]\in \mathrm{Coh}^0_{b,w}(X)$, by Serre duality we have $\mathrm{Hom}(\mathcal{O}_Y,B[i])=\mathrm{Hom}(B,\mathcal{O}_Y(-2)[3-i])=0$ for $i\neq 0,1$. We know $\lim_{(b,w)\to(-1,\frac{1}{2})}\mu^0_{b,w}(B)=+\infty$, so by shrinking the open ball $U'$, we may assume

$$(\mu^0_{b,w})^-(A)>\mu^0_{b,w}(B)>\mu^0_{b,w}(\mathcal{O}_Y(-2)[2]) \tag{18}$$

Then $\sigma^0_{b,w}$-semistability of $B$ and $\mathcal{O}_Y(-2)[2]$ implies that $\mathrm{Hom}(\mathcal{O}_Y,B[1])=\mathrm{Hom}(B,\mathcal{O}_Y(-2)[2])=0$ Moreover, using $E\in\mathcal{K}u(Y)$, we have $\mathrm{Hom}(\mathcal{O}_Y,B)=\mathrm{Hom}(\mathcal{O}_Y,A[1])$. Then (18) gives $\mathrm{Hom}(\mathcal{O}_Y,A[1])=\mathrm{Hom}(A,\mathcal{O}_Y(-2)[2])=0$, so the claim follows. Then Lemma 4.3 implies that $B=\mathcal{Q}_Y[1]=\mathbf{L}_{\mathcal{O}_Y}\mathcal{O}_Y(1)$.

We know $\mathrm{ch}(A)=\mathrm{ch}(\mathcal{O}_Y(-1)[2])$, so $\lim_{(b,w)\to(-1,\frac{1}{2})}Z^0_{b,w}(A)=0$, thus if $A$ is not $\sigma^0_{b,w}$-semistable for any $(b,w)\in U'$, then the destabilizing factors $A_i$ all satisfy $\lim_{(b,w)\to(-1,\frac{1}{2})}\mathrm{Im}[Z^0_{b,w}(A_i)]=0$. Since by (18), we know $\mu^0_{b,w}(A_i)\geq 0$, we have $\mathrm{Re}[Z^0_{b,w}(A_i)]\leq 0$ for all $i$. This implies that $\lim_{(b,w)\to(-1,\frac{1}{2})}\mathrm{Re}[Z^0_{b,w}](A_i)=0$, and so $\mathrm{ch}_{\leq 2}(A_i)$ is a multiple of $\mathrm{ch}_{\leq 2}(\mathcal{O}_Y(-1))$ which is not possible. Thus $A$ is $\sigma^0_{b,w}$-semistable with

$$\mathrm{Hom}(A,\mathcal{O}_Y(-1)[2])=\mathrm{Hom}(\mathcal{O}_Y(1),A[1])=\mathrm{Hom}(\mathcal{O}_Y(1),B)\neq 0.$$

This shows that $A=\mathcal{O}_Y(-1)[2]$ and so $E=i^!\mathcal{Q}_Y[1]$ as $\mathrm{Hom}(\mathcal{Q}_Y[1],\mathcal{O}_Y(-1)[3])=1$ by (11). Finally, Lemma 4.4 implies the stability of $i^!\mathcal{Q}_Y$, which completes the proof. □

**Lemma 4.2.** *Let $F$ be a slope stable sheaf with $\mathrm{ch}_{\leq 2}(F)=(2,-H,sH^2,tH^3)$. Then $s\leq -\frac{1}{2}$. And if $s=-\frac{1}{2}$, then $t\leq\frac{1}{3}$. Moreover, when $s=-\frac{1}{2}$ and $t=\frac{1}{3}$, $F$ is locally free.*

**Proof.** Assume that $s>-\frac{1}{2}$. By [27, Proposition 3.2], we have $s=0$. Thus $\mathrm{ch}_{\leq 2}(F)=\mathrm{ch}_{\leq 2}(F^{\vee\vee})$ and we can assume that $F$ is reflexive. Since $\mathrm{ch}_1^{-1}(F)=1$, there is no wall for $F$ intersects with $b=-1$. Since the line segment connecting $\Pi(F)$ and $\Pi(\mathcal{O}_Y(-2))$ intersects with $b=-1$ inside $\widetilde{U}$, we have $\mathrm{Hom}(F,\mathcal{O}_Y(-2)[1])=H^2(F)=0$. And by the $\mu_H$-stability we have $H^0(F)=0$, which implies $\chi(F)=\frac{c_3(F)+1}{2}<0$. However, since $F$ is reflexive and has rank two, we get $c_3(F)\geq 0$ by [14, Proposition 2.6],[7] which makes a contradiction.

Now we assume that $s=-\frac{1}{2}$. Since there is no wall for $F$ intersects with $b=-1$ and the line segment connecting $\Pi(F)$ and $\Pi(\mathcal{O}_Y(-2))$ intersects with $b=-1$ inside $\widetilde{U}$, we have $\mathrm{Hom}(F,\mathcal{O}_Y(-2)[1])=H^2(F)=0$. Hence by $H^0(F)=0$, we see $\chi(F)=2t-\frac{2}{3}\leq 0$, which implies $t\leq\frac{1}{3}$.

Finally, when $s=-\frac{1}{2}$ and $t=\frac{1}{3}$, we know $F$ is reflexive. Then $c_3(F)=0$, and so $F$ is locally free by [14, Proposition 2.6]. □

**Lemma 4.3.** *Let $F$ be a $\mu_H$-stable sheaf of class $\mathrm{ch}_{\leq 2}(F)=(3,-H,sH^2)$, then $s\leq-\frac{1}{2}$. When $s=-\frac{1}{2}$, we have $\mathrm{ch}_3(F)\leq-\frac{1}{6}H^3$. Moreover, $s=-\frac{1}{2}$ and $\mathrm{ch}_3(F)=-\frac{1}{6}H^3$ if and only if $F=\mathcal{Q}_Y=\mathbf{L}_{\mathcal{O}_Y}\mathcal{O}_Y(1)[-1]$.*

[7] Although [14, Proposition 2.6] only states for $\mathbb{P}^3$, it is well-known that it also works for any smooth projective threefold of Picard rank one.

**Proof.** We know $s \leq -\frac{1}{2}$ from Lemma [27, Proposition 3.2]. When $s = -\frac{1}{2}$, since $\mathrm{ch}_1^{-\frac{1}{2}}(F) = \frac{1}{2}$, and the line segment connecting $\Pi(F)$ and $\Pi(\mathcal{O}_Y(-2))$ intersects $b = -\frac{1}{2}$ inside $\widetilde{U}$, we know that $\mathrm{Hom}(F, \mathcal{O}_Y(-2)[1])) = H^2(F) = 0$. Since $H^0(F) = 0$ by the $\mu_H$-stability of $F$, we see $\chi(F) \leq 0$, which implies $\mathrm{ch}_3(F) \leq -\frac{1}{6}H^3$.

Now assume that $s = -\frac{1}{2}$ and $\mathrm{ch}_3(F) = -\frac{1}{6}H^3$. Then $F$ is reflexive by the previous results. Thus $F[1]$ is $\nu_{0,w}$-semistable for any $w > 0$. Since the line segment connecting $\Pi(F)$ and $\Pi(\mathcal{O}_Y(2))$ intersects with $b = 0$ inside $\widetilde{U}$, we see $\mathrm{Hom}(\mathcal{O}_Y(2), F[1]) = \mathrm{Hom}(F, \mathcal{O}_Y[2]) = 0$. Thus from $\chi(F, \mathcal{O}_Y) = 4$, we see $\hom(F, \mathcal{O}_Y) \geq 4$. Pick four sections and consider the corresponding extension

$$\mathcal{O}_Y^{\oplus 4} \to G \to F[1]$$

Let $\ell$ be the line connecting $\Pi(F)$ and $\Pi(\mathcal{O}_Y)$. We know $G$ is $\nu_{b,w}$-semistable for $(b, w) \in \ell \cap \widetilde{U}$ as $F[1]$ and $\mathcal{O}_Y$ are $\nu_{b,w}$-stable of the same slope. Moreover, $\mathrm{Hom}(\mathcal{O}_Y, F[1]) = 0$. Since $\mathrm{ch}(G) = \mathrm{ch}(\mathcal{O}_Y(1))$, the same argument as in [2, Proposition 4.20] implies that $G \cong \mathcal{O}_Y(1)$. Thus $F \cong \mathcal{Q}_Y$ as $h^0(G) = 4$ and $\mathrm{Hom}(\mathcal{O}_Y, F[1]) = 0$. Note that the $\mu_H$-stability of $\mathcal{Q}_Y$ follows from Lemma 3.3. □

**Lemma 4.4.** *Let $\sigma$ be a Serre-invariant stability condition on $\mathcal{K}u(Y)$. Then $i^!\mathcal{Q}_Y$ is $\sigma$-stable.*

**Proof.** We can assume that $\sigma = \sigma(-\frac{1}{2}, w)$ for some $\frac{1}{4} > w > 0$. As $\mathrm{ch}^{-\frac{1}{2}}(\mathcal{Q}_Y[1]) = \mathrm{ch}^{-\frac{1}{2}}(\mathcal{O}_Y(-1)[1]) = \frac{1}{2}$ is minimal, both $\mathcal{Q}_Y$ and $\mathcal{O}_Y(-1)[1]$ are $\nu_{b=-\frac{1}{2},w}$-stable for any $w > 0$. Then Lemma 2.4 implies that $\mathcal{Q}_Y[1], \mathcal{O}_Y(-1)[2] \in \mathrm{Coh}^0_{b=-\frac{1}{2},w}$ and both are $\sigma^0_{b,w}$-stable. Thus by the exact sequence (12), $i^!\mathcal{Q}_Y[1] \in \mathcal{A}(-\frac{1}{2}, w)$. Suppose for a contradiction that $i^!\mathcal{Q}_Y[1]$ is not $\sigma(-\frac{1}{2}, w)$-semistable, and let $F$ be the destabilizing quotient object of minimum slope. We can write the class $[F] = x\mathbf{v} + y\mathbf{w}$ for $x, y \in \mathbb{Z}$. Then by taking $w = \frac{5}{32}$, one can check the only integers $x, y$ satisfying

$$\mathrm{Im}(Z^0_{-\frac{1}{2},w}(i^!\mathcal{Q}_Y[1])) \geq \mathrm{Im}(Z^0_{-\frac{1}{2},w}(F)) > 0$$

and

$$\mu^0_{-\frac{1}{2},w}(\mathcal{Q}_Y[1]) \leq \mu^0_{-\frac{1}{2},w}(F) < \mu^0_{-\frac{1}{2},w}(i^!\mathcal{Q}_Y[1]) \tag{19}$$

are $(x, y) = (-1, 1)$. The left-hand inequality in (19) comes from the short exact sequence (12) and the fact that $\mu^0_{b=-\frac{1}{2},w}(\mathcal{Q}_Y[1]) < \mu^0_{b=-\frac{1}{2},w}(\mathcal{O}_Y(-1)[2])$ for any $w > 0$. By [38, Theorem 1.1], we know that $F$ fits into a triangle $\mathcal{O}_Y(-1)[1] \to F \to \mathcal{O}_l(-1)$ for a line $l \subset Y$. However $\mathrm{Hom}(i^!\mathcal{Q}_Y[1], F) = \mathrm{Hom}(i^!\mathcal{Q}_Y[1], \mathcal{O}_l(-1)) = 0$, which makes a contradiction. □

**Remark 4.5.** Note that $i^!\mathcal{Q}_Y[1]$ is not stable in double tilted heart $\mathrm{Coh}^0_{b=-\frac{1}{2},w}$. In fact, it is destabilized by $\mathcal{O}_Y(-1)[2]$. There is no wall in the $(b, w)$-plane which would make $i^!\mathcal{Q}_Y[1]$ stable. The objects $E$ fitting in a triangle $\mathcal{Q}_Y[1] \to E[1] \to \mathcal{O}_Y(-1)[2]$ are obtained from triangle (12) as all possible extensions in the other direction. This corresponds to a blow up at the point $[i^!\mathcal{Q}_Y]$ in the Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{K}u(Y), 2\mathbf{v})$ of $\sigma$-stable objects of class $2\mathbf{v}$ in $\mathcal{K}u(Y)$ with the exceptional locus parametrizing those semistable sheaves of rank two, $c_1 = 0, c_2 = 2$ and $c_3 = 0$ not in $\mathcal{K}u(Y)$. For more details, see Section A.

## 5. Moduli spaces on cubic threefolds

In this section, we always fix $Y$ to be a del Pezzo threefold of degree three, i.e. a cubic threefold. The goal of this section is to prove Proposition 5.5 which classifies Bridgeland semistable objects of class $3\mathbf{v}$ in $\mathcal{K}u(Y)$.

Consider the line $\ell_{d=3}$ as defined in section 3.1 which passes through $\Pi(\mathcal{O}_Y(-H))$ and $\Pi(\mathbf{v})$. It is of the equation

$$w = -\frac{5}{6}b - \frac{1}{3},$$

and intersects $\partial\widetilde{U}$ at two points with $b$-values $b_1 = -1$ and $b_2 = -\frac{2}{5}$. We know by Lemma 3.1 that there is no wall for an object $E$ of class $\mathrm{ch}_{\leq 2}(E) = (3, 0, -H^2)$ between the large volume limit ($b < 0$ and $w \gg 0$) and the line $\ell_3$. The following Proposition describes the objects which gets destabilized along the wall $\ell_3$.

**Proposition 5.1.** *Take a point $(b, w) \in \ell_3 \cap \widetilde{U}$ and let $E$ be a strictly $\nu_{b,w}$-semistable object of class $\mathrm{ch}_{\leq 2}(E) = (3, 0, -H^2)$ which is unstable in one side of the wall $\ell_3$. Then the destabilizing sequence is $E_1 \to E \to E_2$ where one of the factors $E_i$ is $\mathcal{O}_Y(-1)[1]$ and the other one $E_j$ is a $\mu_H$-stable sheaf of class $\mathrm{ch}_{\leq 2}(E_j) = (4, -H, -\frac{1}{2}H^2)$. In particular, we have $\mathrm{ch}_3(E) \leq 0$.*

**Proof.** Let $E_1 \to E \to E_2$ be a destabilizing sequence along the wall. If the destabilizing factors $E_1$ and $E_2$ are both sheaves, then $-\frac{2}{5} = b_2 \leq \mu_H(E_i)$ for $i = 1, 2$. Moreover, the location of the wall implies that $\mu_H(E_i) \neq 0$. Thus $\mathrm{ch}_{\leq 1}(E_1) = (3, -H)$ up to relabeling the factors. Moreover $\mathrm{ch}_2(E_1) = -\frac{1}{6}H^2$ because $\Pi(E_1)$ lies on $\ell_3$. We know the wall $\ell_3$ passes through the vertical line $b = -\frac{1}{2}$ at a point inside $\widetilde{U}$, thus $E_1$ is $\nu_{b=-\frac{1}{2},w}$-semistable for some $w > 0$. This implies $E_1$ is $\nu_{b=-\frac{1}{2},w}$-stable for any $w > 0$ by [12, Lemma 3.5], and so $E_1$ is a $\mu_H$-stable sheaf which is not possible by Lemma 5.2. Thus $E_1$ or $E_2$ are not both sheaves.

Let $(r, cH) = \mathrm{ch}_{\leq 1}(\mathcal{H}^{-1}(E_1)) + \mathrm{ch}_{\leq 1}(\mathcal{H}^{-1}(E_2))$, then (8) gives

$$-\frac{2}{5}(r + 3) \leq c \leq -r.$$

Thus either $(r, c)$ is equal to $(2, -2)$ or $(1, -1)$.

**Case I.** First assume $(r, c)$ is equal to $(2, -2)$. We know $\mathcal{H}^{-1}(E_i)$ are torsion-free sheaves. They are even reflexive, otherwise there is a torsion sheaf $T$ supported in co-dimension at least 2 with embedding $T \hookrightarrow \mathcal{H}^{-1}(E_i)[1] \hookrightarrow E_i$ in $\mathrm{Coh}^b(Y)$. This is not possible as $\nu_{b,w}$-slope of semistable factors $E_i$'s are equal to $E$ which is not $+\infty$. Thus one of the following cases can happen:

(a) $\mathrm{ch}_{\leq 1}(\mathcal{H}^{-1}(E_i)) = (1, -H)$ for $i = 1, 2$, or
(b) $\mathcal{H}^{-1}(E_1) = 0$ and $\mathrm{ch}_{\leq 1}(\mathcal{H}^{-1}(E_2)) = (2, -2H)$.

On the other hand, we have

$$\mathrm{ch}_{\leq 1}(\mathcal{H}^0(E_1)) + \mathrm{ch}_{\leq 1}(\mathcal{H}^0(E_2)) = (5, -2H).$$

Since for $i = 1, 2$,

$$\mu_H(\mathcal{H}^0(E_i)) \geq \mu_H^-(\mathcal{H}^0(E_i)) \geq -\frac{2}{5}, \tag{20}$$

the sheaf $\mathcal{H}^0(E_i)$ is torsion supported in dimension at most 1 for either $i = 1$ or $i = 2$.

In case (a), we have $\mathcal{H}^{-1}(E_i) = \mathcal{O}_Y(-1)$ for $i = 1, 2$. By relabeling the factors, we may assume $\mathcal{H}^0(E_2)$ is a torsion sheaf. We know $\Pi(E_2)$ lies on the line $\ell_d$ and

$$\begin{aligned}\mathrm{ch}_{\leq 2}(E_2) &= \mathrm{ch}_{\leq 2}(\mathcal{H}^0(E_2)) - \mathrm{ch}_{\leq 2}(\mathcal{H}^{-1}(E_2)) \\ &= \big(0, 0, \mathrm{ch}_2(\mathcal{H}^0(E_2))\big) - \left(1, -H, \frac{H^2}{2}\right).\end{aligned}$$

This implies that $\mathrm{ch}_2(\mathcal{H}^0(E_2)) = 0$, and so

$$\mathrm{ch}_2(\mathcal{H}^0(E_1)) = \mathrm{ch}_2(\mathcal{H}^{-1}(E_1)) + \mathrm{ch}_2(\mathcal{H}^{-1}(E_2)) + \mathrm{ch}_2(E) = 0$$

which implies $\mathrm{ch}_{\leq 2}(\mathcal{H}^0(E_1)) = (5, -2H, 0)$. Thus $\Pi(\mathcal{H}^0(E_1))$ lies on the boundary of $\widetilde{U}$ which is not possible by [27, Proposition 3.2] as (20) implies that $\mathcal{H}^0(E_1)$ is a $\mu_H$-stable sheaf.

In case (b), we have $E_1 \cong \mathcal{H}^0(E_1)$. Thus $\mathcal{H}^0(E_1)$ cannot be supported in dimension 1, and so $\mathrm{ch}_{\leq 1}(E_1) = \mathrm{ch}_{\leq 1}(\mathcal{H}^0(E_1)) = (5, -2H)$. Since $\Pi(E_1)$ lies on $\ell_d$, we have $\mathrm{ch}_2(E_1) = 0$ which is not again possible by the same argument as in case (a).

**Case II.** Now suppose $(r, c) = (1, -1)$, so by relabeling the factors, we may assume $\mathcal{H}^{-1}(E_1) = 0$ and $\mathcal{H}^{-1}(E_2) = \mathcal{O}_Y(-H)$. Moreover,

$$\mathrm{ch}_{\leq 2}(\mathcal{H}^0(E_1)) + \mathrm{ch}_{\leq 2}(\mathcal{H}^0(E_2)) = \left(4, -H, -\frac{1}{2}H^2\right). \tag{21}$$

Let $\mathrm{ch}_{\leq 2}(E_1) = (r_1, c_1H, s_1H^2)$. Since $\mu_H(\mathcal{H}^0(E_i)) \geq -\frac{2}{5}$, we gain

$$-\frac{2}{5}r_1 \leq \; c_1 \; \leq -\frac{2}{5}r_1 + \frac{3}{5}.$$

Thus $(r_1, c_1)$ is equal to $(0, 0)$, $(1, 0)$, $(3, -1)$, or $(4, -1)$. The first case cannot happen as torsion sheaves supported in dimension $\leq 1$ cannot make a wall. If $(r_1, c_1) = (1, 0)$, then since $\Pi(E_1)$ lies on $\ell_d$, we have $s_1 = -\frac{1}{3}$, thus $E_1$ has the same $\nu_{b,w}$-slope as $E$ with respect to any $(b, w)$, thus it cannot make a wall. If $(r_1, c_1) = (3, -1)$, then $s_1 = -\frac{1}{6}$. We know the wall $\ell_3$ passes through the vertical line $b = -\frac{1}{2}$ at a point inside $\widetilde{U}$, thus [12, Lemma 3.5] implies that $E_1$ is a $\mu_H$-stable sheaf which is not possible by Lemma 5.2. Thus we have

$$\mathrm{ch}_{\leq 2}(E_1) = \left(4, -H, -\frac{1}{2}H^2\right), \tag{22}$$

and $\mathcal{H}^0(E_2)$ is a skyscraper sheaf. Then [2, Proposition 4.20] implies that $E_2 \cong \mathcal{O}_Y(-1)[1]$. Since $E_1$ is $\nu_{b,w}$-semistable on $\ell_3$, it is $\nu_{b=-\frac{1}{2}, w=\frac{1}{12}}$-semistable. Thus by Lemma 5.3, $E_1$ is a $\mu_H$-stable reflexive sheaf. Finally, the last statement follows from Lemma 5.4 that $\mathrm{ch}_3(E_1) \leq -\frac{1}{6}H^3$. □

**Lemma 5.2.** *There is no $\mu_H$-stable sheaf $E$ of class $\mathrm{ch}_{\leq 2}(E) = (3, H, sH^2)$ for $s \geq -\frac{1}{6}$.*

**Proof.** Assume there is such a stable sheaf $E$. By replacing $E$ with its double dual, we may assume $E$ is a reflexive sheaf. Consider the line $\ell$ passing through $\Pi(E)$ and $\Pi(E(-2))$ which is of equation

$$w = -\frac{2}{3}b + \frac{s}{3} + \frac{2}{9}.$$

Since $s \geq -\frac{1}{6}$, it crosses the vertical lines $b = 0$ and $b = -\frac{3}{2}$ at points inside $\widetilde{U}$. Thus [12, Lemma 3.5] implies that both $E$ and $E(-2)[1]$ are $\nu_{b,w}$-stable of the same slope for $(b, w) \in \ell \cap \widetilde{U}$. This implies $\hom(E, E(-2)[1]) = \hom(E, E[2]) = 0$ which is a contradiction as $\hom(E, E) = 1$ and $\chi(E, E) = 18s + 6 \geq 3$. □

**Lemma 5.3.** *Let $b_0 = -\frac{1}{2}$ and pick $w \geq \frac{1}{12}$ (note that the point $(b_0, \frac{1}{12}) \in \widetilde{U} \cap \ell_3$). There is no $\nu_{b_0,w}$-semistable object $E$ of class $\mathrm{ch}_{\leq 2}(E) = (4, -H, sH^2)$ for $s > -\frac{1}{2}$. Moreover, if $s = -\frac{1}{2}$, then $\nu_{b_0,w}$-semistability of $E$ at some $w \geq \frac{1}{12}$ implies that it is $\nu_{b_0,w}$-stable for any $w \geq \frac{1}{12}$. In particular, in this case, $E$ is a $\mu_H$-stable reflexive sheaf.*

**Proof.** Let $E$ be a $\nu_{b_0,w}$-semistable object of class $\mathrm{ch}_{\leq 1}(E) = (4, -H)$ such that $\mathrm{ch}_2(E)H \geq -\frac{H^3}{2}$. We first claim $E$ is $\nu_{b_0,w}$-stable for any $w \geq \frac{1}{12}$. If not, there is a wall $\ell$ for $E$ passing through $\nu_{b_0,w}$ for some $w \geq \frac{1}{12}$. Let $E_1$ be a destabilizing factor of class $(r_1, c_1H, s_1)$ such that $r_1 > 0$. We have

$$0 \;<\; \mathrm{Im}[Z_{b=-\frac{1}{2},w_0}(E_1)] = c_1 + \frac{1}{2}r_1 \;<\; \mathrm{Im}[Z_{b=-\frac{1}{2},w_0}(E_1)] = 1.$$

Thus $c_1 + \frac{1}{2}r_1 = \frac{1}{2}$. If $\frac{c_1}{r_1} < -\frac{2}{5}$, then the position of the wall implies that $\Pi(E_1)$ lies in $\widetilde{U}$ which is not possible. Thus

$$-\frac{2}{5} \leq \frac{c_1}{r_1} = -\frac{1}{2} + \frac{1}{2r_1}$$

which implies $(r_1, c_1)$ is equal to $(3, -1)$, or $(5, -2)$. We know $\Pi(E_1)$ lies above or on the line $\ell_3$. Thus the first cannot happen by Lemma 5.2. In the latter, $s_1 = 0$ and $\Pi(E_1)$ lies on the boundary $\partial\widetilde{U}$ which is not again possible by [27, Proposition 3.2]. Therefore, $E$ is $\nu_{b_0,w}$-stable for $w \geq \frac{1}{12}$ and so a $\mu_H$-stable sheaf.

To complete the proof, we only need to show that we cannot have $s > -\frac{1}{2}$. Assume otherwise, then we may assume $E$ is a reflexive sheaf, so $E(-2)[1]$ is $\nu_{b,w}$-stable for $b > -\frac{9}{4}$ and $w \gg 0$. Since $s \in \frac{1}{6}\mathbb{Z}$, we have $s \geq -\frac{1}{3}$. We know there is no wall for $E(-2)[1]$ crossing the vertical line $b = -2$ for $w > 2$. Thus one can check that $E$ and $E(-2)[1]$ are $\nu_{b,w}$-stable of the same phase for $(b, w) \in \ell \cap \widetilde{U}$ where $\ell$ is the line passing through $\Pi(E)$ and $\Pi(E(-2))$. Hence, $\hom(E, E[2]) = 0$ but $\chi(E, E) \geq 5$, a contradiction. □

**Lemma 5.4.** *Let $E$ be a $\mu_H$-stable sheaf on $Y$ of class*

$$\mathrm{ch}(E) = \left(4, -H, -\frac{1}{2}H^2, sH^3\right).$$

*Then $s \leq -\frac{1}{6}$. Moreover $s = -\frac{1}{6}$ if and only if $E \cong \mathcal{Q}_Y = \mathbf{L}_{\mathcal{O}_Y}\mathcal{O}_Y(1)[-1]$.*

**Proof.** By $\mu_H$-stability of $E$, we have $\mathrm{Hom}(\mathcal{O}_Y, E) = 0 = \mathrm{Hom}(\mathcal{O}_Y, E[3]) = \mathrm{Hom}(E, \mathcal{O}_Y(-2))$. And since the line segment connecting $\Pi(E)$ and $\Pi(\mathcal{O}_Y(-2))$ intersects $b = -\frac{1}{2}$ at a point with $w > \frac{1}{12}$, by Lemma 5.3 we have $0 = \mathrm{Hom}(E, \mathcal{O}_Y(-2)[1]) = \mathrm{Hom}(\mathcal{O}_Y, E[2])$, which gives $\chi(E) = -\hom^1(\mathcal{O}_Y, E) \leq 0$ and $s \leq -\frac{1}{6}$.

Now assume that $s = -\frac{1}{6}$. Then $E$ is reflexive by Lemma 5.3 and the previous result. Thus its shift $E[1]$ is $\nu_{b,w}$-stable for $b > -\frac{1}{4}$ and $w \gg 0$. We know there is no wall for $E[1]$ passing through the vertical line $b = 0$. Therefore $\hom(E, \mathcal{O}_Y[2]) = \hom(\mathcal{O}_Y(2), E[1]) = 0$ and so

$$\hom(E, \mathcal{O}_Y) \geq \chi(E, \mathcal{O}_Y) = 5$$

Hence the first wall $\ell$ for $E[1]$ will be made by $\mathcal{O}_Y[1]$. Pick five linearly independent elements from $\mathrm{Hom}(E, \mathcal{O}_Y)$, and let $G$ be the kernel of the evaluation map in the abelian category of $\nu_{b,w}$-semistable objects of the same slope as $E[1]$ and $\mathcal{O}_Y[1]$ for $(b, w) \in \ell \cap \widetilde{U}$:

$$G \hookrightarrow E[1] \twoheadrightarrow \mathcal{O}_Y^{\oplus 5}[1].$$

We know $\mathrm{ch}(G) = \mathrm{ch}(\mathcal{O}_Y(1))$, so $G \cong \mathcal{O}_Y(1)$ by [2, Proposition 4.20] and the claim follows. □

Finally, we can describe Bridgeland stable objects with class $3\mathbf{v}$ in $\mathcal{K}u(Y)$.

**Proposition 5.5.** *Let $\sigma$ be a Serre-invariant stability condition on $\mathcal{K}u(Y)$ and $E \in \mathcal{K}u(Y)$ be a $\sigma$-(semi)stable object of class $3\mathbf{v}$. Then up to a shift, $E$ is either a Gieseker-(semi)stable sheaf or $i^!\mathcal{Q}_Y$.*

**Proof.** By the uniqueness of Serre-invariant stability conditions on $\mathcal{K}u(Y)$, we can take $\sigma = \sigma(b_0, w_0)$, where $(b_0, w_0) = (-\frac{5}{6}, \frac{13}{36})$. And we can assume $E \in \mathcal{A}(b_0, w_0)$ of class $-3\mathbf{v}$. We have chosen the point $(b_0, w_0) \in V$ so that $\mu^0_{b_0,w_0}(-3\mathbf{v}) = +\infty$. Thus $E$ is $\sigma^0_{b_0,w_0}$-semistable, then Lemma 2.4 implies that $E$ lies in the exact triangle

$$F[1] \to E \to T$$

where $F \in \mathrm{Coh}^{b_0}(Y)$ is $\nu_{b_0,w_0}$-semistable and $T \in \mathrm{Coh}_0(X)$. So we have $\mathrm{ch}(F) = 3\mathbf{v} + \mathrm{ch}(T)$. As the point $(b_0, w_0)$ lies on $\ell_3$, either (i) $F$ is strictly $\nu_{b_0,w_0}$-semistable and unstable above the wall $\ell_3$, or (ii) it is semistable above the line $\ell_3$ and so it's a large volume limit semistable sheaf by Lemma 3.1.

In case (i), Proposition 5.1 implies that $\mathrm{ch}_3(F) \leq 0$ and so $T = 0$. Also combining it with Lemma 5.4 implies that $E[-1] = F$ lies in the non-trivial exact sequence

$$\mathcal{O}_Y(-1)[1] \to E[-1] \to \mathcal{Q}_Y.$$

Since $\mathrm{Hom}(\mathcal{Q}_Y, \mathcal{O}_Y(-1)[2]) = 1$ by (11), we get $E = i^!\mathcal{Q}_Y[1]$.

In case (ii), Lemma 3.1 shows that $F$ is large volume limit semistable and $\mathrm{ch}_3(F) \leq 0$, so $T = 0$. Hence $E[-1] = F$ is a Gieseker-semistable sheaf by Lemma 3.2. □

## 6. Brill–Noether reconstruction

Let $Y \coloneqq Y_d$ be a del Pezzo threefold of Picard rank one of degree $d \geq 2$. In this section, we prove Theorem 1.2 in the introduction in Theorem 6.2.

Let $\mathcal{O}_p$ be the skyscraper sheaf at any point $p \in Y$. We know $\mathbf{L}_{\mathcal{O}_Y(1)}\mathcal{O}_p \cong \mathcal{I}_p(1)[1]$, and so

$$i^*\mathcal{O}_p \cong \mathbf{L}_{\mathcal{O}_Y}(\mathcal{I}_p(1))[1]. \tag{23}$$

We have $\mathrm{ch}(i^*\mathcal{O}_p) = \left(d, -H, -\frac{1}{2}H^2, (\frac{1}{d} - \frac{1}{6})H^3\right) = d\mathbf{v} - \mathbf{w}$. The following proposition characterizes stable objects in $\mathcal{K}u(Y)$ of class $d\mathbf{v} - \mathbf{w}$.

**Proposition 6.1** *([1]). Let $F \in \mathcal{K}u(Y)$ be a $\sigma$-stable object of class $d\mathbf{v} - \mathbf{w}$ for a Serre-invariant stability condition $\sigma$. Then up to a shift, $F$ is either isomorphic to $i^*\mathcal{O}_p$ for a point $p \in Y$, or it is of the form $\mathbf{O}(j_*T)$ where $T$ is a Gieseker-stable reflexive sheaf supported on a hyperplane section $j\colon S \hookrightarrow Y$. This induces a well-defined map*

$$\begin{aligned} \Psi\colon Y &\hookrightarrow \mathcal{M}_\sigma(\mathcal{K}u(Y), d\mathbf{v} - \mathbf{w}) \\ p &\mapsto i^*\mathcal{O}_p \end{aligned} \tag{24}$$

*which gives an embedding of $Y$ into the moduli space $\mathcal{M}_\sigma(\mathcal{K}u(Y), d\mathbf{v} - \mathbf{w})$ as a smooth subvariety.*

**Proof.** Since all stability conditions $\sigma(b, w)$ for $(b, w) \in V$ lie in the same orbit with respect to the action of $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ and they are $\mathbf{O}$-invariant, we can consider $\sigma\left(-\frac{1}{2}, w_0\right)$ where $\left(b = -\frac{1}{2}, w_0\right) \in V$, and characterize $\sigma\left(-\frac{1}{2}, w_0\right)$-stable objects of class $\mathbf{O}^{-1}(d\mathbf{v} - \mathbf{w}) = -\mathbf{w}$.

Take a $\sigma\left(-\frac{1}{2}, w_0\right)$-stable object $E \in \mathcal{A}\left(-\frac{1}{2}, w\right)$ of class $-\mathbf{w}$. Since $\mu^0_{-\frac{1}{2},w}(E) = +\infty$, we know $E$ is $\sigma^0_{-\frac{1}{2},w}$-semistable. Then by [1, Lemma 4.15], $E[-1]$ is $\nu_{-\frac{1}{2},w_0}$-semistable. By the proof of [1, Proposition 4.7], the only wall for $E[-1]$ intersecting $b = -\frac{1}{2}$ is the line $\ell$ passing through $\Pi(\mathcal{O}_Y(-1))$ of slope $-\frac{1}{2}$. Thus when we move up from the point $\left(-\frac{1}{2}, w_0\right)$ along the line $b = -\frac{1}{2}$, either

(i) $E[-1]$ is $\nu_{b=-\frac{1}{2},w}$-semistable for all $w \gg 0$, i.e. it is a Gieseker-stable sheaf, or
(ii) $E[-1]$ gets destabilized along the wall $\ell$.

In case (ii), the destabilizing sequence is of form $A \to E[-1] \to B$, where $\mathrm{ch}_{\leq 2}(B) = \mathrm{ch}_{\leq 2}(\mathcal{O}_Y)$ as in the proof of [1, Proposition 4.7]. Hence $\mathrm{ch}_{\leq 2}(A) = \mathrm{ch}_{\leq 2}(\mathcal{O}_Y(-1)[1])$. Since $\Delta_H(A) = \Delta_H(B) = 0$, $A$ and $B$ are $\nu_{-\frac{1}{2},w}$-semistable for any $w$. This proves $A = \mathcal{O}_Y(-1)[1]$ and $B = \mathcal{I}_p$ for a point $p \in Y$. Thus $E[-1] = E_p$ where $E_p$ is the unique extension

$$\mathcal{O}_Y(-1)[1] \to E_p \to \mathcal{I}_p\,. \tag{25}$$

Thus $\mathbf{O}(E[-1]) = \mathbf{O}(E_p) \cong i^*\,\mathcal{O}_p$ as claimed. Hence $\Psi$ is a well-defined map which is the composition of the embedding $Y \hookrightarrow \mathcal{M}_\sigma(\mathcal{K}u(Y), -\mathbf{w})$ given in [1, Lemma 4.8] (which sends $p \in Y$ to $E_p$), and the isomorphism $\mathcal{M}_\sigma(\mathcal{K}u(Y), -\mathbf{w}) \to \mathcal{M}_\sigma(\mathcal{K}u(Y), d\mathbf{v} - \mathbf{w})$ given by $\mathbf{O}$. In particular, $\Psi$ is an embedding. □

Note that although in [1], $Y$ is assumed to be general, the above results hold for any smooth Fano threefold $Y$ of index 2 and degree $d$. Their aim for the generality assumption is to get an explicit description for Gieseker-stable sheaves of class $\mathbf{w}$ using roots on del Pezzo surfaces, which we do not need in this paper.

**Theorem 6.2** *(Brill–Noether reconstruction for del Pezzo threefolds). Let $\sigma$ be a Serre-invariant stability condition on $\mathcal{K}u(Y)$. Then the map $\Psi$ defined in (24) induces an isomorphism between $Y$ and the* Brill–Noether locus

$$\mathcal{BN}_Y := \{F \in \mathcal{M}_\sigma(\mathcal{K}u(Y), [i^*\,\mathcal{O}_p]) : \ \dim_{\mathbb{C}} \mathrm{Hom}(F,\ i^!\mathcal{Q}_Y) \geq d+1\}$$

*where $\mathcal{O}_p$ is the skyscraper sheaf supported at a point $p \in Y$.*

**Proof.** Recall that $\mathcal{Q}_Y := \mathbf{L}_{\mathcal{O}_Y}\,\mathcal{O}_Y(1)[-1]$ as defined in (9) which is a vector bundle when $d \geq 2$. By adjunction of $i^*$ and $i^!$, we have $\mathrm{RHom}(F,\ i^!\mathcal{Q}_Y) = \mathrm{RHom}(F, \mathcal{Q}_Y)$. Up to a shift, by Proposition 6.1, we can assume $F$ is either (i) isomorphic to $i^*\,\mathcal{O}_p$ for a point $p \in Y$, or (ii) of the form $\mathbf{O}(j_*T)$ where $T$ is a Gieseker-stable sheaf supported on a hyperplane section $j\colon S \hookrightarrow Y$.

In case (i), since $\mathrm{RHom}(\mathcal{O}_Y, \mathcal{Q}_Y) = 0$, by (23), we only need to compute $\mathrm{RHom}(\mathcal{I}_p(1), \mathcal{Q}_Y)$. Since $\mathcal{Q}_Y$ is a bundle of rank $d+1$, we get $\mathrm{RHom}(\mathcal{O}_p, \mathcal{Q}_Y) = \mathbb{C}^{d+1}[-3]$. Now applying $\mathrm{Hom}(-, \mathcal{Q}_Y)$ to the exact sequence $0 \to \mathcal{I}_p(1) \to \mathcal{O}_Y(1) \to \mathcal{O}_p \to 0$, since $\mathrm{RHom}(\mathcal{O}_Y(1), \mathcal{Q}_Y) = \mathbb{C}[-1]$, we see $\mathrm{RHom}(\mathcal{I}_p(1), \mathcal{Q}_Y) = \mathbb{C}[-1] \oplus \mathbb{C}^{d+1}[-2]$. Hence there exists $k \in \mathbb{Z}$, so that $\Psi(p)[k] \in \mathcal{BN}_Y$ for any point $p \in Y$.

In case (ii), by definition of the rotation functor $\mathbf{O}$ in (6), we only need to compute $\mathrm{RHom}(j_*T(1), \mathcal{Q}_Y)$ as $\mathrm{RHom}(\mathcal{O}_Y, \mathcal{Q}_Y) = 0$. Clearly $\mathrm{Hom}(j_*T(1), \mathcal{Q}_Y) = 0$ and

$$\hom\left(j_*T(1), \mathcal{Q}_Y[k]\right) = \hom\left(\mathcal{Q}_Y, j_*T(-1)[3-k]\right) = \hom_S\left(\mathcal{Q}_Y|_S, T(-1)[3-k]\right). \tag{26}$$

Now we apply next Lemma 6.4 to show that the above Hom-spaces vanish for $k = 3, 1$, so we get $\mathrm{RHom}(j_*T(1), \mathcal{Q}_Y) = \mathbb{C}^d[-2]$ as $\chi(j_*T(1), \mathcal{Q}_Y) = d$.

$k = 3$**:** Since $S \in |H|$ is irreducible, Lemma 6.4 implies that both $j_*\,\mathcal{O}_S$ and $j_*\mathcal{Q}_S$ are 2-Gieseker semistable of classes

$$\mathrm{ch}(j_*\,\mathcal{O}_S) = \left(0,\ H,\ -\frac{H^2}{2},\ \frac{H^3}{6}\right) \qquad \text{and} \qquad \mathrm{ch}_{\leq 2}(j_*\mathcal{Q}_S) = \left(0,\ (d+1)H,\ -\frac{d+3}{2}H^2\right).$$

Since $\mathrm{ch}_{\leq 2}(j_*T(-1)) = \left(0, H, -\frac{3}{2}H^2\right)$, comparing slopes implies that

$$\mathrm{Hom}(j_*\,\mathcal{O}_S, j_*T(-1)) = 0 = \mathrm{Hom}(j_*\mathcal{Q}_S, j_*T(-1)).$$

Thus the short exact sequence (27) in Lemma 6.4 below implies that $\mathrm{Hom}(j_*\mathcal{Q}_Y|_S, j_*T(-1)) = 0$.

$k = 1$**:** By Serre-duality on $S$, we know $\hom_S(\mathcal{Q}_Y|_S, T(-1)[2]) = \hom_S(T, \mathcal{Q}_Y|_S)$ which vanishes as

$$\mathrm{Hom}(j_*T, j_*\mathcal{O}_S) = 0 = \mathrm{Hom}(j_*T, j_*\mathcal{Q}_S)$$

by comparing slopes.

Finally, we get $j_*T \notin \mathcal{BN}_Y$ and so $\Psi(Y) = \mathcal{BN}_Y$, then the claim follows from Proposition 6.1. □

**Remark 6.3.** The proof of Theorem 6.2 also shows that $\mathcal{BN}_Y$ can be written as

$$\mathcal{BN}_Y = \{F \in \mathcal{M}_\sigma(\mathcal{K}u(Y), [i^*\mathcal{O}_p]) : \ \mathrm{RHom}(F, \ i^!\mathcal{Q}_Y) \text{ is a two-term complex}\}.$$

**Lemma 6.4.** *Let $Y$ be a del Pezzo threefold of Picard rank one of degree $d \geq 2$, and let $S \hookrightarrow Y$ be a hyperplane section. Then $\mathcal{Q}_Y|_S$ fits into an exact sequence*

$$0 \to \mathcal{O}_S \to \mathcal{Q}_Y|_S \to \mathcal{Q}_S \to 0, \tag{27}$$

*where $\mathcal{Q}_S := \mathbf{L}_{\mathcal{O}_S}\mathcal{O}_S(1)[-1] \in \mathrm{Coh}(S)$ is a $H|_S$-$\mu_H$-semistable bundle on $S$.*

**Proof.** By the restriction of the exact sequence (9), we get the exact sequence

$$0 \to \mathcal{Q}_Y|_S \to \mathcal{O}_S^{\oplus d+2} \to \mathcal{O}_S(1) \to 0$$

on $S$. This gives $\mathrm{RHom}_S(\mathcal{O}_S, \mathcal{Q}_Y|_S) = \mathbb{C}$. Take a non-zero section $s\colon \mathcal{O}_S \to \mathcal{Q}_Y|_S$, then we get the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_S & = & \mathcal{O}_S & \longrightarrow & 0 & & \\
 & & \big\downarrow{\scriptstyle s} & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \mathcal{Q}_Y|_S & \longrightarrow & \mathcal{O}_S^{\oplus d+2} & \longrightarrow & \mathcal{O}_S(1) & \longrightarrow & 0.
\end{array}$$

By taking the cokernel, we get an exact sequence

$$0 \to \mathrm{coker}(s) \to \mathcal{O}_S^{\oplus d+1} \to \mathcal{O}_S(1) \to 0. \tag{28}$$

This implies $\mathrm{coker}(s) \cong \mathcal{Q}_S$ as $\mathrm{RHom}_S(\mathcal{O}_S, \mathrm{coker}(s)) = 0$. To complete the proof, we only need to show $\mathcal{Q}_S$ is $\mu_{H|_S}$-semistable. Assume otherwise, and let $F$ be a destabilizing subsheaf. We may assume $F$ is $\mu_{H|_S}$-stable. Then the exact sequence (28) implies that

$$-\frac{1}{d} = \mu_{H|_S}(\mathcal{Q}_S) < \mu_{H|_S}(F) \leq \mu_{H|_S}(\mathcal{O}_S) = 0.$$

Since $\mathrm{rk}(F) < d$, we must have $\mu_{H|_s}(F) = 0$. We can assume that $F$ is saturated in $\mathcal{Q}_S$, hence is saturated in $\mathcal{O}_S^{d+1}$ as well. By the uniqueness of Jordan–Hölder factors, we get $F \cong \mathcal{O}_S^{\oplus \mathrm{rk}\, F}$. Thus $\mathrm{Hom}_S(\mathcal{O}_S, \mathcal{Q}_S) \neq 0$, which contradicts the construction of $\mathcal{Q}_S$. □

### *6.1. Classical moduli spaces on curves and Brill–Noether reconstruction*

Let $Y$ be a smooth degree 4 del Pezzo threefold, which is the intersection of two quadrics in $\mathbb{P}^5$. There is an FM equivalence $\Phi_{\mathcal{S}}\colon \mathrm{D}^b(C) \xrightarrow{\cong} \mathcal{K}u(Y)$ for a genus two curve $C$. Denote by $M_C(2, \mathcal{L}_1)$ the moduli space of stable vector bundle of rank two with fixed determinant $\mathcal{L}_1$ such that degree $d(\mathcal{L}_1) = 1$. By [35, Theorem 1] we know

$$Y \cong M_C(2, \mathcal{L}_1) \tag{29}$$

Note that $\mathcal{S}$ is the universal spinor bundle on $C \times Y$. On the other hand, using Theorem 6.2 and acting the inverse of the rotation functor $\mathbf{O}$, we get

$$Y \cong \mathbf{O}^{-1}(\mathcal{BN}_Y) = \{E \in \mathcal{M}_\sigma(\mathcal{K}u(Y), -\mathbf{w}) : \ \dim_{\mathbb{C}} \mathrm{Hom}(F,\ i^! \mathcal{O}_Y) \geq 5\}.$$

By [24, Lemma 5.9], $\Phi_{\mathcal{S}}^{-1}(i^! \mathcal{O}_Y) \cong \mathcal{R}[1]$ where $\mathcal{R}$ is a *second Raynaud bundle*, which is a semistable vector bundle of rank 4 and degree 4 on $C$. Moreover, it is unique up to a twist by a line bundle of degree 0, see [24, Section 5.4]. By [1, Section 5.2], the equivalence $\Phi$ sends the Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{K}u(Y), -\mathbf{w})$ to $M_C(2, 1)$. Thus

$$\begin{aligned} Y &\cong \{F \in M_C(2,1) : \ \dim_{\mathbb{C}} \mathrm{Hom}(F,\ \mathcal{R}[1]) \geq 5\} \\ &\cong \{F \in M_C(2,1) : \ \dim_{\mathbb{C}} \mathrm{Hom}(F,\ \mathcal{R}) \geq 1\} \end{aligned} \tag{30}$$

as $\chi(F, \mathcal{R}) = -4$. Comparing (29) and (30) gives the impression that fixing determinant of $F \in M_C(2, 1)$ is equivalent to imposing the Brill–Noether condition.

Let $J(Y)$ be the intermediate Jacobian of $Y$. As in [1, Section 4.4 & Section 5.2],[8] we consider the map

$$\begin{aligned} \mathcal{P} \colon \mathcal{M}_\sigma(\mathcal{K}u(Y), -\mathbf{w}) &\rightarrow J(Y) \\ E &\mapsto \widetilde{c}_2(E) - H^2 \end{aligned}$$

where $\widetilde{c}_2(E)$ is the second Chern class of $E$ up to rational equivalence. We know $\Psi(E_p) = 0$ and $\mathcal{P}^{-1}(0)$ is isomorphic to $Y$, thus $\mathcal{P}([\mathbf{O}(T)]) \neq 0$ where $T$ is a Gieseker-stable sheaf supported on a hyperplane section $S$.

By [1, Section 5.2], $\mathcal{P}^{-1}(0) \cong Y \subset \mathcal{M}_\sigma(\mathcal{K}u(Y), -\mathbf{w})$ such that $Y \cong \{E_p \ : \ p \in Y\}$ (see [1, Proposition 4.7] for definition of $E_p$). Then $Y \cong \mathbf{O}^{-1}(\mathcal{BN}_Y) \cong \mathcal{BN}_Y$.

There is an equivalence $\Phi_1 \colon \operatorname{Pic}^1(C) \to J(Y)$ so that $\Phi_1(\mathcal{L}_1) = 0$ and it induces the commutative diagram [41, Theorem 4.14(c')]

$$\begin{array}{ccc} M_C(2,1) & \xrightarrow{\det} & \operatorname{Pic}^1(C) \\ \big\downarrow{\scriptstyle \Phi_{\mathcal{S}}} & & \big\downarrow{\scriptstyle \Phi_1} \\ \mathcal{M}_\sigma(\mathcal{K}u(Y), -\mathbf{w}) & \xrightarrow{\mathcal{P}} & J(Y). \end{array}$$

This shows that we have an isomorphism

$$M_C(2, \mathcal{L}_1) \cong \ \det{}^{-1}(\mathcal{L}_1) \cong \mathcal{P}^{-1}(0) \ \cong \mathcal{BN}_Y.$$

[8] Although some parts in [1] assume $Y$ to be general, the arguments in [1, Section 4.4 & Section 5.2] do not use any generic assumption.

## 7. Uniqueness of the gluing object

In this section, we prove the following Theorem.

**Theorem 7.1.** *Let $\Phi\colon \mathcal{K}u(Y) \simeq \mathcal{K}u(Y')$ be an exact equivalence of Kuznetsov components of del Pezzo threefolds of the same degree $d$ where $2 \leq d \leq 4$.*

(i) *If $d = 2, 3$, there exist a unique pair of integers $m_1, m_2 \in \mathbb{Z}$ with $0 \leq m_1 \leq 3$ when $d = 2$ and $0 \leq m_1 \leq 5$ when $d = 3$, so that*

$$\Phi(i^! \mathcal{Q}_Y) \cong \mathbf{O}^{m_1}(i'^! \mathcal{Q}_{Y'})[m_2].$$

(ii) *If $d = 4$, there exists a unique pair of integers $m_1, m_2$ and a unique auto-equivalence $T_{\mathcal{L}_0} \in \mathrm{Aut}^0(\mathcal{K}u(Y'))$ (see Section 7.3 for definition) so that*

$$\Phi(i^! \mathcal{Q}_Y) \cong \mathbf{O}^{m_1} \circ T_{\mathcal{L}_0}(i'^! \mathcal{Q}_{Y'})[m_2].$$

*Here $i'\colon \mathcal{K}u(Y') \hookrightarrow \mathrm{D}^b(Y')$ is the inclusion functor.*

**Remark 7.2.** Theorem 7.1 also holds if we replace $i^! \mathcal{Q}_Y$ and $i'^! \mathcal{Q}_{Y'}$ by $i^! \mathcal{O}_Y$ and $i'^! \mathcal{O}_{Y'}$, respectively. The reason is that $\mathbf{O}(i^! \mathcal{O}_Y) \cong i^! \mathcal{Q}_Y$ and the proof only uses the properties of Bridgeland moduli spaces with respect to Serre-invariant stability conditions and objects in them, which are all preserved by $\mathbf{O}$.

**Remark 7.3.** The proof of Theorem 7.1 actually shows that if $\Phi$ maps $\mathbf{v}$ and $\mathbf{w}$ to $\mathbf{v}'$ and $\mathbf{w}'$ respectively, then $\Phi(i^! \mathcal{Q}_Y) = i'^! \mathcal{Q}_{Y'}$ up to shift and action of $T_{\mathcal{L}_0}$ (when $d = 4$).

We first discuss the action of equivalences on the numerical Grothendieck groups, and then investigate each degree separately.

**Lemma 7.4.** *Let $Y$ and $Y'$ be two del Pezzo threefolds of Picard rank ones of degree $d$ and $\Phi : \mathcal{K}u(Y) \to \mathcal{K}u(Y')$ an equivalence. Let $\phi : \mathcal{N}(\mathcal{K}u(Y)) \to \mathcal{N}(\mathcal{K}u(Y'))$ be the induced isometry. Then*

(a) *If $\phi(m\mathbf{v}) = m\mathbf{v}'$ for a non-zero integer $m$, then $\phi(\mathbf{v}) = \mathbf{v}'$ and $\phi(\mathbf{w}) = \mathbf{w}'$.*
(b) *Up to composing with $\mathbf{O}$ and $[1]$, $\phi$ maps classes $\mathbf{v}$ and $\mathbf{w}$ to $\mathbf{v}'$ and $\mathbf{w}'$, respectively.*

**Proof.** Recall that the numerical Grothendieck group $\mathcal{N}(\mathcal{K}u(Y'))$ has no torsion. In part (a), from $\phi(m\mathbf{v}) = m\mathbf{v}'$ we have $\phi(\mathbf{v}) = \mathbf{v}'$. Now we assume that $\phi(\mathbf{w}) = a\mathbf{v}' + b\mathbf{w}'$ for $a, b \in \mathbb{Z}$. Using $\chi(\mathbf{v}, \mathbf{w}) = -1$ and $\chi(\mathbf{w}, \mathbf{v}) = 1 - d$, we get $\chi(\mathbf{v}', a\mathbf{v}' + b\mathbf{w}') = -1$ and $\chi(a\mathbf{v}' + b\mathbf{w}', \mathbf{v}') = 1 - d$. Thus we obtain $-a - b = -1$ and $-a + (1 - d)b = 1 - d$, which gives $(a, b) = (0, 1)$ when $d \neq 2$. When $d = 2$, using $\chi(\mathbf{w}, \mathbf{w}) = \chi(a\mathbf{v}' + b\mathbf{w}', a\mathbf{v}' + b\mathbf{w}') = -d$, we obtain $(a, b) = (0, 1)$ or $(2, -1)$. We claim the latter cannot happen, otherwise

$$\phi(\mathbf{v}) = \mathbf{v}' \qquad \text{and} \qquad \phi(\mathbf{v} - \mathbf{w}) = -(\mathbf{v}' - \mathbf{w}').$$

For any line $l \subset Y$, we define $\mathcal{J}_l := \mathbf{O}^{-1}(\mathcal{I}_l)[1] \in \mathcal{K}u(Y)$ as in [38]. Fix two lines $l_1, l_2 \subset Y$ such that $l_1 \cap l_2 \neq \varnothing$. Then by [38, Remark 4.8], we have $\mathrm{Hom}(\mathcal{I}_{l_1}, \mathcal{J}_{l_2}) \neq 0$. Since $\chi(\mathcal{I}_{l_1}, \mathcal{J}_{l_2}) = 0$ and $\mathrm{Hom}(\mathcal{I}_{l_1}, \mathcal{J}_{l_2}[n]) = 0$ when $n \leq -1$ and $n \geq 2$, we get $\mathrm{Hom}(\mathcal{I}_{l_1}, \mathcal{J}_{l_2}[1]) \neq 0$.

Let $\sigma$ be a Serre-invariant stability condition on $\mathcal{K}u(Y)$, then by [38, Theorem 1.1] any $\sigma$-stable object of class $[\mathcal{I}_l]$ in $\mathcal{K}u(Y)$ is the shifted ideal sheaf $\mathcal{I}_{l'}[k]$ for some line $l'$ on $Y$. The same claim also holds for objects of class $[\mathcal{J}_l] = -[\mathbf{O}^{-1}(\mathcal{I}_l)]$ as $\sigma$ is $\mathbf{O}$-invariant. Recall that there is a unique Serre-invariant stability condition on $\mathcal{K}u(Y)$ up to $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action. Since $\Phi$ commutes with the Serre functor, $\Phi.\sigma$ is also a Serre-invariant stability condition on $\mathcal{K}u(Y')$. Thus by $\phi(\mathbf{v}) = \mathbf{v}'$ and $\phi(\mathbf{v}-\mathbf{w}) = -(\mathbf{v}'-\mathbf{w}')$, up to a shift, we can assume that $\Phi(\mathcal{I}_{l_1}) = \mathcal{I}_{l_1'}$ and $\Phi(\mathcal{J}_{l_2}) = \mathcal{J}_{l_2'}[k]$ for lines $l_1', l_2' \subset Y'$ and an odd integer $k$. Thus we get $\mathrm{Hom}(\mathcal{I}_{l_1'}, \mathcal{J}_{l_2'}[k]) = \mathrm{Hom}(\mathcal{I}_{l_1'}, \mathcal{J}_{l_2'}[1+k]) \neq 0$. This implies $k = 0$ and makes a contradiction which completes the proof of part (a).

For part (b), we claim that up to composing with $\mathbf{O}$ and [1], $\mathbf{v}$ maps to $\mathbf{v}'$. Indeed, the image of $\mathbf{v}$ is still a $(-1)$-class in $\mathcal{N}(\mathcal{K}u(Y))$ since $\Phi$ is an equivalence. Then the claim for $d \geq 3$ follows from [32, Corollary 4.2]. And up to sign, a $(-1)$-class is either $\mathbf{v}'$ or $\mathbf{v}'-\mathbf{w}'$ for $d = 2$, and $\mathbf{v}', \mathbf{w}'$ or $\mathbf{v}'-\mathbf{w}'$ for $d = 1$. They are permuted by rotation functor $\mathbf{O}$ and the claim follows. Thus the result follows from part (a) and the claim above. □

### 7.1. Degree 2 case

We first consider a del Pezzo threefold $Y$ of degree 2 which is a quartic double solid. It is a double cover $\pi : Y \to \mathbb{P}^3$ which is ramified over a smooth surface $R \subset \mathbb{P}^3$ of degree 4. The branch divisor of $\pi$ maps isomorphic to $R$, which we also denote by $R \subset Y$. The involution on $Y$ given by the double cover is denoted by $\tau$. The Serre functor of $\mathcal{K}u(Y)$ is $S_{\mathcal{K}u(Y)} = \tau[2]$. Moreover we have $\mathcal{O}_Y(R) = \mathcal{O}_Y(2)$. The key idea to prove Theorem 7.1 is to investigate the singular locus of a suitable moduli space in $\mathcal{K}u(Y)$.

**Lemma 7.5.** *Let $\sigma$ be a Serre-invariant stability condition on $\mathcal{K}u(Y)$. Then the singular locus of the moduli space $\mathcal{M}_\sigma(\mathcal{K}u(Y), 2\mathbf{v}-\mathbf{w})$ is at least two dimensional, consists of objects of form $i^*\mathcal{O}_p$ such that $p \in R$, and $\mathbf{O}(j_*F)$ where $j\colon S \hookrightarrow Y$ is a hyperplane section and $F$ is a reflexive sheaf on $S$ with $\tau(j_*F) \cong j_*F$.*

**Proof.** Since $\sigma$ is $\mathbf{O}$-invariant, the functor $\mathbf{O}$ makes an isomorphism $\mathcal{M}_\sigma(\mathcal{K}u(Y), -\mathbf{w}) \cong \mathcal{M}_\sigma(\mathcal{K}u(Y), 2\mathbf{v} - \mathbf{w})$). Thus for any $F \in \mathcal{M}_\sigma(\mathcal{K}u(Y), 2\mathbf{v}-\mathbf{w})$, there exists $E \in \mathcal{M}_\sigma(\mathcal{K}u(Y), -\mathbf{w})$ so that $F = \mathbf{O}(E)$. Since $\mathrm{RHom}(F, F) = \mathrm{RHom}(E, E)$, we only need to consider the smoothness of $[E]$ in $\mathcal{M}_\sigma(\mathcal{K}u(Y), -\mathbf{w})$. By Proposition 6.1 and its proof, there are two possibilities:
Case (i). $E = E_p$ for a point $p \in Y$ as defined in (25). Since $\tau(E_p) = E_{\tau(p)}$, we know that $[E]$ is a singular point if and only if $\mathrm{Ext}^2(E_p, E_p) = \mathrm{Hom}(E_p, E_{\tau(p)}) \neq 0$, which is equivalent to $p = \tau(p)$, i.e. $p \in R$.
Case (ii). $E = j_*F$ is a reflexive Gieseker-stable sheaf supported on a hyperplane section $j\colon S \hookrightarrow Y$. Then by $\sigma$-stability, $\mathrm{Ext}^2(E, E) = \mathrm{Hom}(E, \tau E) \neq 0$ if and only if $\tau(j_*F) \cong j_*F$. □

The next Proposition analyzes further the second case in Lemma 7.5.

**Proposition 7.6.** *Let $\sigma$ be a Serre-invariant stability condition and $j_*F \in \mathcal{K}u(Y)$ be a $\sigma$-stable object of class $\mathbf{w}$, where $j\colon S \hookrightarrow Y$ is a hyperplane section and $F$ is a reflexive sheaf on $S$. Let $E \in \mathcal{K}u(Y)$ be a Gieseker-stable sheaf of class $2\mathbf{v}$. Assume that $\tau(j_*F) \cong j_*F$, then we have*

$$\mathrm{RHom}(\mathbf{O}(j_*F), E) = \mathbb{C}^2[-2].$$

**Proof.** By Lemma 3.2, $E$ is 2-Gieseker-stable. Thus $j^*E$ is a sheaf by the torsion-freeness of $E$. Since $F \in \mathcal{K}u(Y)$, we see $\mathrm{RHom}(\mathbf{O}(j_*F), E) = \mathrm{RHom}(j_*F(1), E)$. It is clear that $\mathrm{Hom}(j_*F(1), E) = 0$.

We claim $\mathrm{Ext}^3(j_*F(1), E) = \mathrm{Hom}(E, j_*F(-1)) = 0$. If not, there is a nonzero map $\pi\colon E \to j_*F(-1)$ with $\mathrm{ch}_{\leq 1}(\ker(\pi)) = (2, -H)$ and $H.\mathrm{ch}_2(\ker(\pi)) \geq 1$. Thus by [27, Proposition 3.2], $\ker(\pi)$ cannot be $\mu_H$-semistable. But since it is torsion-free, it has a two-term HN filtration $E_1 \hookrightarrow \ker(\pi) \twoheadrightarrow E_2$. Since $E_1$ is a subsheaf of $E$ as well, we have $\mathrm{ch}_{\leq 2}(E_1) = (1, 0, \frac{a}{2}H^2)$ where $a \leq -2$. Thus $\mathrm{ch}(E_2) = (1, -H)$ and $\mathrm{ch}_2(E_2).H = \mathrm{ch}_2(\ker(\pi)).H - a \geq 3$, which is not possible.

Therefore we get $-\operatorname{ext}^1(\mathbf{O}(j_*F), E) + \operatorname{ext}^2(\mathbf{O}(j_*F), E) = \chi(\mathbf{O}(j_*F), E) = 2$, so we only need to show $\operatorname{Ext}^1(\mathbf{O}(j_*F), E) = 0$. Note that

$$\operatorname{Ext}^1(\mathbf{O}(j_*F), E) = \operatorname{Ext}^1(j_*F(1), E) = \operatorname{Hom}_S(F, j^*E) = \operatorname{Hom}(j_*F, j_*j^*E).$$

Assume there is a non-zero map $s \in \operatorname{Hom}_S(F, j^*E)$. Since $F$ is torsion-free of rank one on $S$, $s$ is injective. Let $G := \operatorname{coker}(s)$.

**Claim:** $G$ is a torsion-free sheaf on $S$. As $G$ has rank one on $S$, this implies $j_*G$ is Gieseker-stable. To this end, we consider a commutative diagram of exact triangles

$$\begin{array}{ccccc} 0 & \longrightarrow & j_*F & = & j_*F \\ \downarrow & & \downarrow{\scriptstyle j_*(s)} & & \downarrow \\ E & \longrightarrow & j_*j^*E & \longrightarrow & E(-1)[1] \end{array}$$

By taking cones, we get a commutative diagram with rows and columns exact

$$\begin{array}{ccccc} 0 & \longrightarrow & j_*F & = & j_*F \\ \downarrow & & \downarrow{\scriptstyle j_*(s)} & & \downarrow \\ E & \longrightarrow & j_*j^*E & \longrightarrow & E(-1)[1] \\ \| & & \downarrow & & \downarrow \\ E & \xrightarrow{a} & j_*G & \longrightarrow & K[1] \end{array}$$

Here $K$ is a sheaf since it is an extension of $j_*F$ and $E(-1)$ from the construction. Thus $a$ is surjective and $K = \ker(a)$. Note that $\operatorname{ch}(K) = 2\mathbf{v} - \mathbf{w}$. We consider two cases:

— If $K$ is $\mu_H$-stable, by Lemma 4.2 $K$ is locally free. Since $E$ is torsion-free, we get torsion-freeness of $G$ on $S$.

— If $K$ is not $\mu_H$-semistable, then there is a destabilizing sequence $K_1 \to K \to K_2$ where both $K_1$ and $K_2$ are rank one $\mu_H$-stable sheaf. Note that since $K$ is a subsheaf of $E$, it is torsion-free. The composition of injections $K_1 \to K \to E$ and 2-Gieseker stability of $E$ implies that $\operatorname{ch}_{\leq 2}(K_1) = (1, 0, -\frac{a+2}{2}H^2)$ where $a \geq 0$. Since $K_2$ is torsion-free with class $\operatorname{ch}_{\leq 2}(K_2) = (1, -H, \frac{1+a}{2}H^2)$, we get $a = 0$. Thus $K_2 \cong \mathcal{I}_{p_2}(-H)$ for some points $p_2$ on $Y$. We denote $W := \operatorname{coker}(K_1 \hookrightarrow E)$. Then we have a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & K_1 & = & K_1 & \longrightarrow & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & j_*G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & K_2 & \longrightarrow & W & \longrightarrow & j_*G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

with rows and columns exact. Since $\operatorname{RHom}(\mathcal{O}_Y, j_*F) = \operatorname{RHom}(\mathcal{O}_Y, j_*j^*E) = 0$, we get the vanishing $\operatorname{RHom}(\mathcal{O}_Y, j_*G) = 0$. In particular, $G$ has no zero-dimensional torsion. We know $\operatorname{ch}_{\leq 2}(W) = (1, 0, 0)$, from 2-Gieseker-stability of $E$, we see that the torsion part of $W$ is zero-dimensional, which is not possible as $G$ has no zero-dimensional subsheaf. Thus $W \cong \mathcal{I}_p$ for some points $p$ in $Y$, so the third row in the above diagram gives the short exact sequence $\mathcal{I}_{p_2}(-H) \hookrightarrow \mathcal{I}_p \twoheadrightarrow j_*G$ which implies $j_*G$ is pure.

Hence $G$ is torsion-free as claimed. Thus $j^*E$ is also torsion-free as $F$ and $G$ are.

We divide the rest of the proof into two cases.

**Case 1.** First assume $E$ is not locally free. By Proposition A.4, we have an exact sequence

$$0 \to E \to \mathcal{O}_Y^{\oplus 2} \to Q \to 0,$$

where $Q$ is supported on a curve. Hence we get a triangle $j^*E \to \mathcal{O}_S^{\oplus 2} \to j^*Q$ on $S$. Since $Q$ is supported on a curve, $\mathcal{H}^i_{\mathrm{Coh}(S)}(j^*Q)$ is at most one-dimensional for each $i$ by [15, Lemma 3.29]. Using the fact that $j^*E$ is torsion-free, we see $j^*Q \in \mathrm{Coh}(S)$ and hence $j^*E \subset \mathcal{O}_S^{\oplus 2}$. Thus $F \subset \mathcal{O}_S^{\oplus 2}$, which implies that $\mathrm{Hom}_S(F, \mathcal{O}_S) = \mathrm{Hom}(j_*F, j_*\mathcal{O}_S) \neq 0$. Hence $\mathrm{Hom}(j_*F, \mathcal{O}_Y(-1)[1]) \neq 0$, which contradicts $j_*F \in \mathcal{K}u(Y)$.

**Case 2.** Now assume $E$ is locally free, and so $j^*E$ is locally free. Then taking $\mathcal{H}om_S(-, F)$ from the short exact sequence $F \to j^*E \to G$ gives $\mathcal{E}xt^1_S(F, F) = \mathcal{E}xt^2_S(G, F)$. By Lemma 7.8, we get $\mathcal{E}xt^2_S(G, F) \neq 0$, which implies $\mathrm{Ext}^3(j_*G, j_*F) \neq 0$ from Lemma 7.7. However, by Serre duality we get $\mathrm{Hom}(j_*F, j_*G(-2)) \neq 0$, which contradicts the Gieseker-stability of $j_*F$ and $j_*G$. $\square$

**Lemma 7.7.** *Let $j\colon S \hookrightarrow Y$ be a hyperplane section and $E, F$ be two coherent sheaves on $S$ with $E$ torsion-free. Let $n \geq 2$ be the maximal integer with $\mathcal{E}xt^n_S(E, F) \neq 0$. Then $\mathrm{Ext}^{n+1}(j_*E, j_*F) \neq 0$.*

**Proof.** We first show that any hyperplane section $S \in |\mathcal{O}_Y(1)|$ is normal and Gorenstein. Since $Y$ is Gorenstein, $S$ is too. Then by Serre's criterion, to prove the normality of $S$, we only need to prove $S$ has only finitely many singular closed points. Note that $S = \pi^{-1}(P)$ is a double cover ramified over $R \cap P$ for a projective plane $P \subset \mathbb{P}^3$. By the property of double cover, we only need to show $R \cap P$ has isolated singularities. This follows from applying [26, Corollary 3.4.19] to $R$.

Since $S$ is normal, the non-locally free locus of $E$ has codimension two. Thus $\mathcal{E}xt^i_S(E, F)$ is supported on points for any $i > 0$. Now we compute $\mathcal{E}xt^i_Y(j_*E, j_*F) := \mathcal{H}^i(R\mathcal{H}om_Y(j_*E, j_*F))$. By adjunction, we have

$$R\mathcal{H}om_Y(j_*E, j_*F) = j_*R\mathcal{H}om_S(j^*j_*E, F).$$

Since $\mathcal{H}^0(j^*j_*E) \cong E$ and $\mathcal{H}^{-1}(j^*j_*E) \cong E(-1)$, using [15, (3.8)], we have a spectral sequence convergent to $\mathcal{E}xt^{p+q}_S(j^*j_*E, F)$ with $E_2^{p,0} = \mathcal{E}xt^p_S(E, F)$, $E_2^{p,1} = \mathcal{E}xt^p_S(E, F)(1)$ and $E_2^{p,q} = 0$ for $p \neq 0, 1$. Therefore, we see that $\mathcal{E}xt^i_S(j^*j_*E, F)$ is supported on points for $i \geq 2$. Moreover, the term $E_2^{n,1}$ survives, hence $E_2^{n,1} = E_\infty^{n,1} \neq 0$ implies that $\mathcal{E}xt^{n+1}_S(j^*j_*E, F) \neq 0$. Thus $\mathcal{E}xt^i_Y(j_*E, j_*F)$ is supported on $S$, and furthermore supported on points for $i \geq 2$ with $\mathcal{E}xt^{n+1}_Y(j_*E, j_*F) \neq 0$.

Next, using [15, (3.16)], we have a spectral sequence

$$E_2^{p,q} = H^p(\mathcal{E}xt^q_Y(j_*E, j_*F)) \Rightarrow \mathrm{Ext}^{p+q}(j_*E, j_*F).$$

By the previous argument, we know that $E_2^{0,n+1} = \mathrm{length}(\mathcal{E}xt^{n+1}_Y(j_*E, j_*F)) \neq 0$. Moreover, from the dimension of support, we see $E_2^{p,q} = 0$ for $p \in \{1, 2\}, q \geq 2$ and any $p \geq 3$, $q \in \mathbb{Z}$. Since $n \geq 2$, this implies $E_2^{0,n+1} = E_\infty^{0,n+1} \neq 0$, which gives $\mathrm{Ext}^{n+1}(j_*E, j_*F) \neq 0$. $\square$

**Lemma 7.8.** *Let $\sigma$ be a Serre-invariant stability condition on $\mathcal{K}u(Y)$ and $j_*F \in \mathcal{K}u(Y)$ be a $\sigma$-stable object where $j\colon S \hookrightarrow Y$ is a hyperplane section and $F$ is a reflexive sheaf on $S$. If $\tau(j_*F) \cong j_*F$, or equivalently $\mathrm{Ext}^2(j_*F, j_*F) \neq 0$, then $\mathcal{E}xt^1_S(F, F)$ is non-zero and supported on a single point with length one.*

**Proof.** Note that by $\sigma$-stability and $S_{\mathcal{K}u(Y)} = \tau[2]$, we know that $\mathrm{Ext}^2(j_*F, j_*F) = \mathrm{Hom}(j_*F, \tau(j_*F)) \neq 0$ if and only if $\mathrm{Ext}^2(j_*F, j_*F) = \mathrm{Hom}(j_*F, \tau(j_*F)) = \mathbb{C}$.

Since $F$ is reflexive and $S$ is normal, we have $\mathcal{H}om_S(F, F) = \mathcal{O}_S$. Moreover, by Lemma 7.7 and the vanishing $\mathrm{Ext}^i(j_*F, j_*F) = 0$ when $i \geq 3$, we get $\mathcal{E}xt^i_S(F, F) = 0$ for $i \geq 2$. Therefore, if we compute $\mathcal{E}xt^2_Y(j_*F, j_*F)$ as in Lemma 7.7, we get $\mathcal{H}om_Y(j_*F, j_*F) = j_*\mathcal{O}_S$, $\mathcal{E}xt^1_Y(j_*F, j_*F)$ is an extension of $j_*\mathcal{O}_S(1)$ with $j_*\mathcal{E}xt^1_S(F, F)$, and $\mathcal{E}xt^2_Y(j_*F, j_*F) = j_*\mathcal{E}xt^1_S(F, F)(1)$. Thus, if we compute $\mathrm{Ext}^i(j_*F, j_*F)$ as in Lemma 7.7, we see $\mathrm{Ext}^2(j_*F, j_*F) = H^0(\mathcal{E}xt^2_Y(j_*F, j_*F))$. This implies that $\mathcal{E}xt^1_S(F, F)$ is non-zero and supported on a single point with length one. $\square$

**Proof of Theorem 7.1 for degree** $d = 2$**.** Note that $\mathbf{O}^4 \cong [2]$ when $d = 2$, so by Lemma 7.4 we can assume that there is a pair of integers $m_1, \delta$ with $0 \leq m_1 \leq 3$ and $\delta = 0, 1$ such that $\mathbf{O}^{-m_1} \circ \Phi[\delta]$ maps classes $\mathbf{v}$ and $\mathbf{w}$ on $Y$ to $\mathbf{v}'$ and $\mathbf{w}'$ on $Y'$, respectively. Moreover, we know such $m_1$ and $\delta$ is unique by looking at the action of $\mathbf{O}$ and $[1]$ on $\mathcal{N}(\mathcal{K}u(Y))$ and using the restricted values of $m_1$ and $\delta$. We may replace $\Phi$ by $\mathbf{O}^{-m_1} \circ \Phi[\delta]$.

We know $\Phi(i^!\mathcal{Q}_Y) \in \mathcal{M}_{\Phi(\sigma)}(\mathcal{K}u(Y'), 2\mathbf{v}')$, so by Proposition 4.1, up to a shift, it is either $i'^!\mathcal{Q}_{Y'}$ or a Gieseker-stable sheaf $E'$. Assume for a contradiction that the latter happens. We know $\Phi$ maps the singular locus of $\mathcal{M}_\sigma(\mathcal{K}u(Y), 2\mathbf{v} - \mathbf{w})$ to the singular locus of $\mathcal{M}_{\Phi(\sigma)}(\mathcal{K}u(Y'), 2\mathbf{v}' - \mathbf{w}')$.

- Assume that $\Phi$ maps $R \subset \mathcal{M}_\sigma(\mathcal{K}u(Y), 2\mathbf{v} - \mathbf{w})$ to $R' \subset \mathcal{M}_{\Phi(\sigma)}(\mathcal{K}u(Y'), 2\mathbf{v}' - \mathbf{w}')$. Thus by Proposition 6.1, we get $\mathrm{RHom}\left(\Phi(i^*\mathcal{O}_p), \Phi(i^!\mathcal{Q}_Y)\right)$ and so $\mathrm{RHom}\left(i'^*\mathcal{O}_{p'}, E'\right) = \mathrm{RHom}\left(\mathcal{O}_{p'}, E'\right)$ are a two-term complex for all $p' \in R$. But this makes a contradiction since $E'$ is torsion-free so the non-locally free locus of $E'$ has at most dimension one.
- Assume that $\Phi$ does not map $R \subset \mathcal{M}_\sigma(\mathcal{K}u(Y), 2\mathbf{v} - \mathbf{w})$ to $R' \subset \mathcal{M}_{\Phi(\sigma)}(\mathcal{K}u(Y'), 2\mathbf{v}' - \mathbf{w}')$. By Lemma 7.5, there is a point $p \in R$ such that $\Phi(i^*\mathcal{O}_p) = \mathbf{O}(j_*F)$ up to shift, where $j\colon S \hookrightarrow Y'$ is a hyperplane section and $F$ is a reflexive sheaf on $S$ with $\tau'(j_*F) \cong j_*F$. Moreover, $\mathrm{RHom}(i^*\mathcal{O}_p, i^!\mathcal{Q}_Y) = \mathrm{RHom}(\mathbf{O}(j_*F), E')$ is a two-term complex. But this contradicts Proposition 7.6.

Hence in both cases, we get $\Phi(i^!\mathcal{Q}_Y) = i'^!\mathcal{Q}_{Y'}[m_2 + \delta]$ for a unique $m_2 \in \mathbb{Z}$ and the claim follows. $\square$

**Remark 7.9.** [1, Lemma 4.4] claims $\mathrm{Ext}^2(j_*F, j_*F) = 0$ for any hyperplane section $j\colon S \hookrightarrow Y$ and a rank one reflexive sheaf $F$ on $S$ such that $j_*F \in \mathcal{K}u(Y)$. However, the proof is valid only for smooth $S$ via the vanishing of $\mathcal{E}xt^1_S(F, F)$. That is why in this section, we investigated further the singular locus in order to prove Theorem 7.1.

### *7.2. Degree three case*

Now assume $Y$ is a cubic threefold.

**Proof of Theorem 7.1 for degree** $d = 3$**.** In this case $\mathbf{O}^6 \cong [4]$, so by Lemma 7.4 there is a unique pair of integer $m_1, \delta$ with $0 \leq m_1 \leq 5$ and $\delta = 0, 1$ such that $\mathbf{O}^{-m_1} \circ \Phi[\delta]$ maps classes $\mathbf{v}$ and $\mathbf{w}$ on $Y$ to $\mathbf{v}'$ and $\mathbf{w}'$ on $Y'$, respectively. We replace $\Phi$ by $\mathbf{O}^{-m_1} \circ \Phi[\delta]$. Then by Proposition 5.5, the object $\Phi(i^!\mathcal{Q}_Y) \in \mathcal{M}_{\Phi(\sigma)}(\mathcal{K}u(Y'), 3\mathbf{v}')$, up to a shift, is either $i'^!\mathcal{Q}_{Y'}$ or a Gieseker-semistable sheaf $E'$. Assume for a contradiction that the latter happens.

By [2, Lemma 7.5, Theorem 8.7], $\mathcal{BN}_{Y'}$ is the union of all rational curves in $\mathcal{M}_{\Phi(\sigma)}(\mathcal{K}u(Y'), 3\mathbf{v}' - \mathbf{w}')$. Thus $\phi(\mathcal{BN}_Y) = \mathcal{BN}_{Y'}$. In other words, for any $p \in Y$ we have $\Phi(i^*\mathcal{O}_p) \cong i'^*\mathcal{O}_{p'}$ for a point $p' \in Y'$ up to shift and vice versa. In particular, $\mathrm{RHom}(i'^*\mathcal{O}_{p'}, E')$ is a two-term complex for all $p' \in Y'$. But this contradicts the torsion-freeness of $E'$. Hence we get $\Phi(i^!\mathcal{Q}_Y) = i'^!\mathcal{Q}_{Y'}[m_2 + \delta]$ for a unique $m_2 \in \mathbb{Z}$ as claimed. $\square$

### 7.3. Degree four case

Let $Y$ be a del Pezzo threefold of degree 4, then $\mathcal{K}u(Y)$ is equivalent to the bounded derived category $\mathrm{D}^b(C)$ of a smooth projective curve $C$ of genus 2. As in [24, Section 5], we fix the Fourier–Mukai equivalence $\Psi_{\mathcal{S}}\colon \mathrm{D}^b(C) \to \mathcal{K}u(Y)$ for the universal spinor bundle $\mathcal{S}$ on $C \times Y$, where we see $Y$ as a moduli space of stable rank 2 bundles on $C$ with fixed determinant $\xi$ of degree $\deg(\xi) = 1$.

For any line bundle $\mathcal{L}$ on $C$, we denote the induced auto-equivalence of $\mathcal{K}u(Y)$ by $T_{\mathcal{L}} := \Psi_{\mathcal{S}} \circ (- \otimes \mathcal{L}) \circ \Psi_{\mathcal{S}}^{-1}$. We write $\mathrm{Aut}^0(\mathcal{K}u(Y))$ for the subgroup of $\mathrm{Aut}(\mathcal{K}u(Y))$ consisting of $T_{\mathcal{L}}$ such that $\mathcal{L} \in \mathrm{Pic}^0(C)$. We will apply the following two facts about the action of $\mathbf{O}$:

(a) By [24, Lemma 5.2], we know that via the equivalence $\Psi_{\mathcal{S}}$, the action of $\mathbf{O}$ on $\mathcal{N}(\mathcal{K}u(Y))$ is the same as twisting by a degree $-1$ line bundle on $C$, up to sign.
(b) Since any stability condition $\sigma$ on $\mathcal{K}u(Y)$ is $\mathbf{O}$-invariant, (semi)stability of a vector bundle on $C$ will be preserved after the action of $\Psi_{\mathcal{S}}^{-1} \circ \mathbf{O} \circ \Psi_{\mathcal{S}}$.

**Proof of Theorem 7.1 for degree $d = 4$.** By Lemma 7.4, there exist a pair of integers $m_1, m_2$ such that $\mathbf{O}^{-m_1} \circ \Phi[-m_2]$ maps classes $\mathbf{v}$ and $\mathbf{w}$ to $\mathbf{v}'$ and $\mathbf{w}'$. By the above point (a), such $m_1$ is unique. Furthermore, we can take $m_2$ uniquely by imposing the condition that $\Psi_{\mathcal{S}}^{-1} \circ (\mathbf{O}^{-m_1} \circ \Phi[-m_2]) \circ \Psi_{\mathcal{S}}\colon \mathrm{D}^b(C) \to \mathrm{D}^b(C')$ maps bundles to bundles. We replace $\Phi$ by $\mathbf{O}^{-m_1} \circ \Phi[-m_2]$.

By [24, Lemma 5.9], $\Psi_{\mathcal{S}}^{-1}(i^! \mathcal{O}_Y)$ is a second Raynaud bundle[9] $\mathcal{R}$ on $C$ up to a shift. We know this bundle is unique on $C$ up to tensoring by a line bundle of degree zero. Thus by the above point (b), $\Psi_{\mathcal{S}}^{-1}(\mathbf{O}(i^! \mathcal{O}_Y)) = \Psi_{\mathcal{S}}^{-1}(i^! \mathcal{Q}_Y)$ is also unique up to tensoring by a line bundle of degree zero. Indeed, let $R$ and $R'$ be two Raynaud bundles, then we can assume $R' = R \otimes L_0$ for a degree 0 line bundle $L_0$. Note that $\mathbf{O} = f_* \circ (- \otimes L_{-1})$ for a degree $-1$ line bundle $L_{-1}$ up to shift, so that $\mathbf{O}(R') = \mathbf{O}(R \otimes L_0) = f_*(R) \otimes L'_{-1}$ a degree $-1$ line bundle $L'_{-1}$. On the other hand, $\mathbf{O}(R) = f_*(R) \otimes L''_{-1}$ for a degree $-1$ line bundle $L''_{-1}$. Hence $\mathbf{O}(R)$ and $\mathbf{O}(R')$ differ by a degree 0 line bundle. This proves there is a unique line bundle $\mathcal{L}_0$ on $C'$ such that

$$(\Psi'^{-1}_{\mathcal{S}'} \circ \Phi(i^! \mathcal{Q}_Y)) \otimes \mathcal{L}_0^{-1} = \Psi'^{-1}_{\mathcal{S}'}(i'^! \mathcal{Q}_{Y'})$$

and so the claim follows. □

Proof of Theorem 7.1, in particular, implies the following.

**Corollary 7.10.**

— *If $d = 2$, $i^! \mathcal{Q}_Y$ is the unique object in the moduli space $\mathcal{M}_\sigma(\mathcal{K}u(Y), 2\mathbf{v})$ satisfying the following condition: there is a 2-dimensional sub-locus $\mathcal{M}'$ of the singular locus of the moduli space $\mathcal{M}_\sigma(\mathcal{K}u(Y), 2\mathbf{v} - \mathbf{w})$ such that for any object $\mathcal{E} \in \mathcal{M}'$, $\mathrm{RHom}(\mathcal{E}, i^! \mathcal{Q}_Y)$ is a two-term complex.*
— *If $d = 3$, $i^! \mathcal{Q}_Y$ is the unique object in the moduli space $\mathcal{M}_\sigma(\mathcal{K}u(Y), 3\mathbf{v})$ such that for any object $\mathcal{E} \in \mathcal{M}_\sigma(\mathcal{K}u(Y), 3\mathbf{v} - \mathbf{w})$ whose corresponding point lies on a rational curve in $\mathcal{M}_\sigma(\mathcal{K}u(Y), 3\mathbf{v} - \mathbf{w})$, $\mathrm{RHom}(\mathcal{E}, i^! \mathcal{Q}_Y)$ is a two-term complex.*
— *If $d = 4$, $i^! \mathcal{Q}_Y$ is a unique object in the moduli space $\mathcal{M}_\sigma(\mathcal{K}u(Y), 4\mathbf{v})$ up to the action of an auto-equivalence $T_{\mathcal{L}} \in \mathrm{Aut}^0(\mathcal{K}u(Y))$ such that $\mathrm{RHom}(\mathcal{E}, i^! \mathcal{Q}_Y)$ is a two-term complex for any object $\mathcal{E} \in \mathcal{M}_\sigma(\mathcal{K}u(Y), -3\mathbf{v} + \mathbf{w})$.*

[9] It is a semistable vector bundle of rank 4 and degree 4 on a genus 2 curve so that for any line bundle $\mathcal{L}$ of degree zero on $C$, we have $\mathrm{Hom}(\mathcal{L}, \mathcal{R}) \neq 0$.

**Proof.** The first two cases for degree $d = 2, 3$ are a direct result of proof of Theorem 7.1. For $d = 4$, note that $\chi(\mathcal{E}, i^! \mathcal{Q}_Y) = 0$. Combining [24, Lemma 5.9] and [38, Theorem 1.1] implies that $i^! \mathcal{O}_Y$ is a unique object in $\mathcal{M}_\sigma(\mathcal{K}u(Y), 4(\mathbf{v} - \mathbf{w}))$ up to the action of an auto-equivalence $T_{\mathcal{L}_0} \in \mathrm{Aut}^0(\mathcal{K}u(Y))$ such that $\mathrm{RHom}(\mathcal{E}, i^! \mathcal{Q}_Y)$ is a two-term complex for any object $\mathcal{E} \in \mathcal{M}_\sigma(\mathcal{K}u(Y), \mathbf{v})$. Thus taking the rotation functor $\mathbf{O}$ implies the claim. □

*7.4. Categorical Torelli theorem*

As a result of Theorem 7.1, we show a categorical Torelli theorem for any del Pezzo threefold of degree $2 \leq d \leq 4$.

**Corollary 7.11.** *Let $Y$ and $Y'$ be del Pezzo threefolds of degree $2 \leq d \leq 4$ such that $\Phi : \mathcal{K}u(Y) \simeq \mathcal{K}u(Y')$ is an exact equivalence of Kuznetsov components, then $Y \cong Y'$.*

**Proof.** By Theorem 7.1, we can assume that $\Phi(i^! \mathcal{Q}_Y) \cong i'^! \mathcal{Q}_{Y'}$. There is an isometry of numerical Grothendieck group $\phi : \mathcal{N}(\mathcal{K}u(Y)) \cong \mathcal{N}(\mathcal{K}u(Y'))$ induced by $\Phi : \mathcal{K}u(Y) \simeq \mathcal{K}u(Y')$. As $\Phi(i^! \mathcal{Q}_Y) \cong i'^! \mathcal{Q}_{Y'}$, we get $\phi(\mathbf{v}) = \mathbf{v}'$ and $\phi(\mathbf{w}) = \mathbf{w}'$ by Lemma 7.4. Then the result follows from the uniqueness of Serre-invariant stability conditions and Theorem 6.2 via the same argument in [19, Corollary 6.11]. □

## 8. Auto-equivalences of Kuznetsov components of del Pezzo threefolds

In this section, we are going to prove Theorem 8.2 and Corollary 8.4, and describe the auto-equivalences of Kuznetsov components of del Pezzo threefolds. We begin with a lemma.

**Lemma 8.1.** *Let $f, g \colon Y \to Y'$ be two isomorphisms between del Pezzo threefolds of Picard one. If $f_*|_{\mathcal{K}u(Y)} = g_*|_{\mathcal{K}u(Y)} \colon \mathcal{K}u(Y) \to \mathcal{K}u(Y')$, then $f = g$. Thus the homomorphism*

$$\mathrm{Aut}(Y) \to \mathrm{Aut}(\mathcal{K}u(Y)), \quad f \mapsto f_*|_{\mathcal{K}u(Y)}$$

*is injective.*

**Proof.** We know $f_*$ and $g_*$ maps $\mathcal{O}_Y$ and $\mathcal{O}_Y(1)$ to $\mathcal{O}_{Y'}$ and $\mathcal{O}_{Y'}(1)$ respectively. For any point $p \in Y$, we know $f_*(\mathcal{O}_p) = \mathcal{O}_{f(p)}$ and the same for $g$. Thus we have

$$f_*(i^* \mathcal{O}_p) = i'^* \mathcal{O}_{f(p)} \qquad \text{and} \qquad g_*(i^* \mathcal{O}_p) = i'^* \mathcal{O}_{g(p)}\,.$$

Since $f_*|_{\mathcal{K}u(Y)} = g_*|_{\mathcal{K}u(Y)}$, we get $i'^* \mathcal{O}_{f(p)} = i'^* \mathcal{O}_{g(p)}$, i.e. $i'^* \mathcal{O}_{f(p)}$ and $i'^* \mathcal{O}_{g(p)}$ correspond to the same point in the moduli space $\mathcal{M}_\sigma(\mathcal{K}u(Y), d\mathbf{v} - \mathbf{w})$ by Proposition 6.1. Thus the embedding $\Psi$ in (24) implies that $f(p) = g(p)$ for any point $p \in Y$. Since both $Y$ and $Y'$ are smooth, we get $f = g$. □

**Theorem 8.2.** *Let $Y$ and $Y'$ be two del Pezzo threefolds of the same degree $d$ where $d = 2, 3$ or $4$, and let $\Phi \colon \mathcal{K}u(Y) \to \mathcal{K}u(Y')$ be an exact equivalence of Fourier–Mukai type such that $\Phi(i^! \mathcal{Q}_Y) = i'^! \mathcal{Q}_{Y'}$. Then $\Phi = f_*|_{\mathcal{K}u(Y)}$ for a unique isomorphism $f \colon Y \to Y'$.*

**Proof.** Since $[i^! \mathcal{Q}_Y] = d\mathbf{v} \in \mathcal{N}(\mathcal{K}u(Y))$, Lemma 7.4 (a) implies that $\Phi$ maps $\mathbf{v}$ and $\mathbf{w}$ to $\mathbf{v}'$ and $\mathbf{w}'$, respectively. Then Theorem 6.2 shows that for any $p \in Y$, there is a point $p' \in Y'$ such that

$$\Phi(i^* \mathcal{O}_p) \cong i'^* \mathcal{O}_{p'}\,. \tag{31}$$

Conversely, for any $p' \in Y'$, there is $p \in Y$ such that the above holds. From Remark 7.2, we also have $\Phi(i^! \mathcal{O}_Y) = i'^! \mathcal{O}_{Y'}$. Thus using [29, Proposition 2.5 & Remark 2.2], $\Phi$ can be extended to an equivalence $\mathcal{O}_Y(1)^\perp \cong \mathcal{O}_{Y'}(1)^\perp$, denoted again by $\Phi$, so that $\Phi(\mathcal{O}_Y) \cong \mathcal{O}_{Y'}$. Since $i^* = \mathbf{L}_{\mathcal{O}_Y}\mathbf{L}_{\mathcal{O}_Y(1)}$, (31) implies that $\Phi(\mathbf{L}_{\mathcal{O}_Y(1)}(\mathcal{O}_p)) \cong \mathbf{L}_{\mathcal{O}_{Y'}(1)}(\mathcal{O}_{p'})$.

Let $j : \mathcal{O}_Y(1)^\perp \hookrightarrow \mathrm{D}^b(Y)$ and $j' : \mathcal{O}_{Y'}(1)^\perp \hookrightarrow \mathrm{D}^b(Y')$ be the natural inclusions. We know

$$j^! \mathcal{O}_Y(1) = \mathbf{R}_{\mathcal{O}_Y(-1)}(\mathcal{O}_Y(1)),$$

so it lies in the triangle

$$\mathcal{O}_Y(-1)[2] \to j^! \mathcal{O}_Y(1) \to \mathcal{O}_Y(1). \tag{32}$$

The next step is to compute $i^*(j^! \mathcal{O}_Y(1)) = \mathbf{L}_{\mathcal{O}_Y}(j^! \mathcal{O}_Y(1))$. Using the triangle above, it is easy to see $\mathrm{RHom}(\mathcal{O}_Y, j^! \mathcal{O}_Y(1)) = \mathbb{C}^{d+2}$, so we have an triangle

$$\mathcal{O}_Y^{\oplus d+2} \to j^! \mathcal{O}_Y(1) \to \mathbf{L}_{\mathcal{O}_Y}(j^! \mathcal{O}_Y(1)). \tag{33}$$

Thus by taking cohomology we obtain

$$\mathcal{O}_Y(-1)[2] \to \mathbf{L}_{\mathcal{O}_Y}(j^! \mathcal{O}_Y(1)) \to \mathcal{Q}_Y[1]$$

and so $\mathbf{L}_{\mathcal{O}_Y}(j^! \mathcal{O}_Y(1)) = i^! \mathcal{Q}_Y[1]$. Therefore, we know that $\Phi(\mathbf{L}_{\mathcal{O}_Y}(j^! \mathcal{O}_Y(1))) = \mathbf{L}_{\mathcal{O}_{Y'}}(j'^! \mathcal{O}_{Y'}(1))$. Applying $\Phi$ to (33) gives a triangle

$$\mathcal{O}_{Y'}^{\oplus d+2} \to \Phi(j^! \mathcal{O}_Y(1)) \to i'^! \mathcal{Q}_{Y'}[1]. \tag{34}$$

This implies that $\mathcal{H}^{-2}(\Phi(j^! \mathcal{O}_Y(1))) = \mathcal{O}_{Y'}(-1)$ and we have the long exact sequence

$$0 \to \mathcal{H}^{-1}(\Phi(j^! \mathcal{O}_Y(1))) \to \mathcal{Q}_{Y'} \to \mathcal{O}_{Y'}^{\oplus d+2} \to \mathcal{H}^0(\Phi(j^! \mathcal{O}_Y(1))) \to 0. \tag{35}$$

Since $j^! \mathcal{O}_Y(1) \in \mathcal{O}_Y(1)^\perp$, by the adjunction of mutations, we have $\mathrm{RHom}(\mathbf{L}_{\mathcal{O}_Y(1)}(\mathcal{O}_p), j^! \mathcal{O}_Y(1)) = \mathrm{RHom}(\mathcal{O}_p, j^! \mathcal{O}_Y(1))$ for any $p \in Y$. Thus we have

$$\begin{aligned}\mathrm{RHom}(\mathcal{O}_p, j^! \mathcal{O}_Y(1)) &= \mathrm{RHom}(\mathbf{L}_{\mathcal{O}_Y(1)}(\mathcal{O}_p), j^! \mathcal{O}_Y(1)) = \mathrm{RHom}(\Phi(\mathbf{L}_{\mathcal{O}_Y(1)}(\mathcal{O}_p)), \Phi(j^! \mathcal{O}_Y(1))) \\ &= \mathrm{RHom}(\mathbf{L}_{\mathcal{O}_{Y'}(1)}(\mathcal{O}_{p'}), \Phi(j^! \mathcal{O}_Y(1))) = \mathrm{RHom}(\mathcal{O}_{p'}, \Phi(j^! \mathcal{O}_Y(1))).\end{aligned}$$

Using (32), we know that $\mathrm{RHom}(\mathcal{O}_p, j^! \mathcal{O}_Y(1)) = \mathbb{C}[-1] \oplus \mathbb{C}[-3]$. Hence $\mathrm{RHom}(\mathcal{O}_{p'}, \Phi(j^! \mathcal{O}_Y(1))) = \mathbb{C}[-1] \oplus \mathbb{C}[-3]$ for any $p' \in Y'$. By Serre-duality, we have

$$\mathrm{RHom}(\Phi(j^! \mathcal{O}_Y(1)), \mathcal{O}_{p'}) = \mathbb{C} \oplus \mathbb{C}[-2]. \tag{36}$$

Then from [4, Proposition 5.4], $\Phi(j^! \mathcal{O}_Y(1))$ is quasi-isomorphic to a complex

$$A_{-2} \to A_{-1} \xrightarrow{\alpha} A_0, \tag{37}$$

where $A_k$ is a bundle of rank $r_k$ sitting in degree $k$ in the complex. Note that (37) is a locally-free resolution of $\Phi(j^! \mathcal{O}_Y(1))$. Therefore, we have $\mathcal{H}^0(\Phi(j^! \mathcal{O}_Y(1))) \cong \mathrm{coker}(\alpha)$ and by applying $\mathrm{Hom}(-, \mathcal{O}_{p'})$ to (37), we have a complex

$$\mathrm{Hom}(A_0, \mathcal{O}_{p'}) = \mathbb{C}^{r_0} \xrightarrow{\overline{\alpha}} \mathrm{Hom}(A_{-1}, \mathcal{O}_{p'}) \to \mathrm{Hom}(A_{-2}, \mathcal{O}_{p'}).$$

Since $\mathrm{Hom}(\Phi(j^! \mathcal{O}_Y(1)), \mathcal{O}_{p'}) = \mathbb{C}$, we get $\ker(\overline{\alpha}) = \mathbb{C}$. But note that $\overline{\alpha}$ can be factored as $\mathrm{Hom}(A_0, \mathcal{O}_{p'}) \to \mathrm{Hom}(\mathrm{im}(\alpha), \mathcal{O}_{p'}) \hookrightarrow \mathrm{Hom}(A_{-1}, \mathcal{O}_{p'})$ which implies

$$\hom((\mathrm{im}(\alpha), \mathcal{O}_{p'})) \geq r_0 - 1.$$

Since $p' \in Y'$ is an arbitrary closed point, we have $\mathrm{rk}(\mathrm{im}(\alpha)) \geq r_0 - 1$. Thus $\mathrm{rk}(\mathcal{H}^0(\Phi(j^! \mathcal{O}_Y(1)))) \leq 1$. Since $\mathcal{H}^0(\Phi(j^! \mathcal{O}_Y(1)))$ sits in an exact sequence (35) and $\mathrm{rk}(\mathcal{Q}_{Y'}) = d + 1$, we have $\mathrm{rk}(\mathcal{H}^0(\Phi(j^! \mathcal{O}_Y(1)))) = 1$, which implies

$$\mathrm{rk}(\mathcal{H}^{-1}(\Phi(j^! \mathcal{O}_Y(1)))) = 0.$$

Since $\mathcal{Q}_{Y'}$ is torsion-free, we have $\mathcal{H}^{-1}(\Phi(j^! \mathcal{O}_Y(1))) = 0$ and $\mathcal{H}^0(\Phi(j^! \mathcal{O}_Y(1))) = \mathcal{O}_{Y'}(1)$ by definition (9). Thus $\Phi(j^! \mathcal{O}_Y(1))$ lies in the exact triangle

$$\mathcal{O}_{Y'}(-1)[2] \to \Phi(j^! \mathcal{O}_Y(1)) \to \mathcal{O}_{Y'}(1). \tag{38}$$

Note that $\mathrm{Hom}(\Phi(j^! \mathcal{O}_Y(1)), \Phi(j^! \mathcal{O}_Y(1))) = \mathrm{Hom}(j^! \mathcal{O}_Y(1), j^! \mathcal{O}_Y(1)) = \mathrm{Hom}(j^! \mathcal{O}_Y(1), \mathcal{O}_Y(1)) = \mathbb{C}$ by (32), so the exact triangle (38) is non-splitting. Since $\mathrm{Hom}(\mathcal{O}_{Y'}(1), \mathcal{O}_{Y'}(-1)[3]) = 1$, we get

$$\Phi(j^! \mathcal{O}_Y(1)) \cong j^! \mathcal{O}_{Y'}(1).$$

Then applying again [29, Proposition 2.5] shows that the equivalence $\Phi\colon \mathcal{O}_Y(1)^\perp \to \mathcal{O}_{Y'}(1)^\perp$ can be extended to an equivalence $\Phi\colon \mathrm{D}^b(Y) \xrightarrow{\cong} \mathrm{D}^b(Y')$ such that $\Phi(\mathcal{O}_Y(1)) \cong \mathcal{O}_{Y'}(1)$. Then [15, Corollary 5.23] implies that $\Phi$ is the composition of $f_*$ for an isomorphism $f\colon Y \to Y'$ with the twist by a line bundle on $Y$. Since we know $\Phi(\mathcal{O}_Y) = \mathcal{O}_{Y'}$, we get $\Phi = f_*$. Finally, such isomorphism $f$ is unique by Lemma 8.1. □

**Remark 8.3.** Combining Theorem 7.1 with Theorem 8.2 provides an alternative proof of *Categorical Torelli theorem* for del Pezzo threefold of degree $2 \leq d \leq 4$.

As an application, we obtain a complete description of the group $\mathrm{Aut}_{\mathrm{FM}}(\mathcal{K}u(Y))$ of exact auto-equivalences of $\mathcal{K}u(Y)$ of Fourier–Mukai type.

**Corollary 8.4.** *Let $Y$ be a del Pezzo threefold of Picard rank one and degree $d$, and $\Phi \in \mathrm{Aut}_{\mathrm{FM}}(\mathcal{K}u(Y))$ be an auto-equivalence of $\mathcal{K}u(Y)$ of Fourier–Mukai type.*

(i) *If $2 \leq d \leq 3$, there exist a unique $f \in \mathrm{Aut}(Y)$ and unique pair of integers $m_1, m_2 \in \mathbb{Z}$ with $0 \leq m_1 \leq 3$ when $d = 2$ and $0 \leq m_1 \leq 5$ when $d = 3$, so that*

$$\Phi = \mathbf{O}^{m_1} \circ f_*|_{\mathcal{K}u(Y)} \circ [m_2].$$

(ii) *If $d = 4$, there exists a unique $f \in \mathrm{Aut}(Y)$ and unique pair of integers $m_1, m_2$ and a unique auto-equivalence $T_{\mathcal{L}_0} \in \mathrm{Aut}^0(\mathcal{K}u(Y))$ (see Section 7.3 for definition) so that*

$$\Phi = \mathbf{O}^{m_1} \circ T_{\mathcal{L}_0} \circ f_*|_{\mathcal{K}u(Y)} \circ [m_2].$$

**Proof.** The result follows from Theorem 7.1 and Theorem 8.2. □

**Remark 8.5.** Assume $Y' = Y$, then Remark 7.3 and Theorem 8.2 show that the homomorphism

$$\mathrm{Aut}(Y) \to \mathrm{Aut}_{\mathrm{FM}}(\mathcal{K}u(Y)), \quad f \mapsto f_*|_{\mathcal{K}u(Y)}$$

is injective, and its image together with [2] generates the sub-group of auto-equivalences that act trivially on $\mathcal{N}(\mathcal{K}u(Y))$. This strengthens [21, Lemma B.2.3].

**Remark 8.6.** For a del Pezzo threefold $Y$ of degree 5, its Kuznetsov component $\mathcal{K}u(Y)$ is equivalent to the derived category of representations of 3-Kronecker quiver. It is known the group of auto-equivalences of $\mathcal{K}u(Y)$ is $\mathbb{Z} \times (\mathbb{Z} \rtimes \mathrm{PGL}_3(\mathbb{C}))$ by [34, Theorem 4.3].

**Remark 8.7.** For index one prime Fano threefold of genus 6 and 8, one can apply similar techniques in this section to compute the group of auto-equivalences of their Kuznetsov components. Combining with the results for del Pezzo threefold of degree 2 and 3, we can identify the group of automorphisms of cubic threefold and correspondent genus 8 prime Fano threefolds and provide another disproof of Kuznetsov's Fano threefold conjecture ([23, Conjecture 3.7]). For details, we refer interested readers to the arxiv version of our paper.

## Appendix A. Moduli space of instanton sheaves on quartic double solids

In this section, we fix $Y$ to be a quartic double solid and study the moduli space $M_Y(2,0,2)$ of semistable sheaves of rank two, $c_1 = 0, c_2 = 2, c_3 = 0$ and the Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{K}u(Y), 2\mathbf{v})$ of semistable objects of class $2\mathbf{v}$ in the Kuznetsov component $\mathcal{K}u(Y)$.

### *A.1. Classifications*

As is shown in Proposition 4.1 that up to shift, the $\sigma$-stable objects of class $2\mathbf{v}$ in the Kuznetsov component $\mathcal{K}u(Y)$ of a quartic double solid $Y$ is either a two-term complex $i^!\mathcal{Q}_Y$ or a Gieseker semistable sheaf of rank two, $c_1 = 0, c_2 = 2$ and $c_3 = 0$. Denote by $E$ such a sheaf. It is clear that $H^1(Y, E(-1)) = 0$ since $E \in \mathcal{K}u(Y)$. Then it is an instanton sheaf in the sense of [32, Definition 6.2]. To study the geometric structure and properties of the Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{K}u(Y), 2\mathbf{v})$, first we classify sheaves in the moduli space $M_Y^{inst}(2,0,2)$ of instanton sheaves on $Y$.

**Proposition A.1.** *Let $E \in M_Y(2,0,2)$. Then $E \notin \mathcal{K}u(Y)$ if and only if it is a locally free sheaf fitting into an exact sequence*

$$0 \to \mathcal{O}_Y(-1) \to \mathcal{Q}_Y \to E \to 0. \tag{39}$$

*If $E \in \mathcal{K}u(Y)$, then $E$ is*

(1) *either a strictly Gieseker-semistable sheaf, which is an extension of two ideal sheaves of lines,*
(2) *or a non-locally free sheaf fitting into a short exact sequence*

$$0 \to E \to \mathcal{O}_Y^{\oplus 2} \to Q \to 0,$$

*where $Q = \theta_C(1)$ is the theta characteristic of a smooth conic $C$, or $Q$ is a sheaf on a codimension two linear section $C$ of $Y$ given by*

$$0 \to \mathcal{O}_C \to Q \to R \to 0,$$

*where $R$ is a zero-dimensional sheaf on $C$ of length two,*

(3) *or a $\mu_H$-stable vector bundle that $E(1)$ is globally generated and fits into the short exact sequence*

$$0 \to \mathcal{O}_Y(-H) \to E \to I_D(H) \to 0,$$

*where $D$ is the zero locus of a generic section of $H^0(E(1))$, which is a degree 4 smooth elliptic curve.*

**Proof.** If $E$ is strictly Gieseker-semistable, then the result follows from applying Lemma 3.1 to Jordan–Hölder factors. If $E$ is Gieseker-stable, the result follows from Proposition A.4, Lemma A.8 and Proposition A.9 below. □

In the following, we are going to prove the results that will be used in the proof of Proposition A.1.

**Lemma A.2.** *Let $E$ be a $\mu_H$-semistable reflexive sheaf of rank two, $c_1(E) = 0$ and $H^0(E)$=0. Then $E$ is $\mu_H$-stable.*

**Proof.** If not, its Jordan–Hölder filtration with respect to $\mu_H$-stability has two terms $E_1 \hookrightarrow E \twoheadrightarrow E_2$ where $E_1$ and $E_2$ are $\mu_H$-stable sheaves with $\mathrm{ch}_{\leq 1}(E_i) = (1, 0)$. Then $E_1^{\vee\vee} = \mathcal{O}_Y$ since $\mathrm{Pic}(Y) = \mathbb{Z}H$. Then taking the double dual, we get a non-zero map $\mathcal{O}_Y \to E^{\vee\vee} = E$, which contradicts $H^0(E) = 0$. □

**Lemma A.3.** *There is no $\mu_H$-semistable reflexive sheaf $E$ of classes*

(1) $\mathrm{ch}(E) = (2, 0, -\frac{1}{2}H^2, \alpha_1 H^3)$,
(2) $\mathrm{ch}(E) = (2, 0, -H^2, \alpha_2 H^3)$ *where* $\alpha_2 \neq 0$, *and*
(3) $\mathrm{ch}(E) = (2, 0, 0, \alpha_3 H^3)$ *where* $\alpha_3 \neq 0$.

*Moreover, if* $\mathrm{ch}(E) = 2\mathrm{ch}(\mathcal{O}_Y)$, *then* $E \cong \mathcal{O}_Y^{\oplus 2}$.

**Proof.** Note that being rank two and reflexive implies $c_3(E) \geq 0$ by [14, Proposition 2.6], hence $\alpha_i \geq 0$. Then the case (2) follows from Lemma A.2 and Lemma 3.1. And case (3) follows from the same argument as in [2, Proposition 4.20].

So we only need to prove (1). Assume for a contradiction that $E$ is a $\mu_H$-semistable reflexive sheaf of classes $\mathrm{ch}(E) = (2, 0, -\frac{1}{2}H^2, \alpha_1 H^3)$ with $\alpha_1 \geq 0$. We know that there is no wall for $E$ crossing the vertical line $b = -\frac{1}{2}$, so $\mathrm{Hom}(E, \mathcal{O}_Y(-2)[1]) = H^2(E) = 0$. And by $\mu_H$-semistability, we get $H^0(E) = H^3(E) = 0$, which implies

$$2\alpha_1 + 1 = \chi(\mathcal{O}_Y, E) = -\hom(\mathcal{O}_Y, E[1]) \leq 0$$

which makes a contradiction. If $\mathrm{ch}(E) = 2\mathrm{ch}(\mathcal{O}_Y)$, then

$$\hom(\mathcal{O}_Y, E) - \hom(\mathcal{O}_Y, E[1]) = 2.$$

Thus Jordan–Hölder factors of $E$ with respect to the $\mu_H$-stability are all $\mathcal{O}_Y$, and the result follows. □

**Proposition A.4.** *Let $E \in \mathcal{K}u(Y)$ be a non-reflexive Gieseker-stable sheaf of character $2\mathbf{v}$, then $E$ fits into a short exact sequence*

$$0 \to E \to \mathcal{O}_Y^{\oplus 2} \to Q \to 0,$$

*where $Q = \theta_C(1)$ is the theta characteristic of a smooth conic $C$, or $Q$ is a sheaf on a codimension two linear section $C$ of $Y$ given by*

$$0 \to \mathcal{O}_C \to Q \to R \to 0,$$

*where $R$ is a zero-dimensional sheaf on $C$ of length two.*

**Proof.** Taking the reflexive hull of $E$ gives the exact sequence

$$E \to E^{\vee\vee} \to Q \tag{40}$$

where $E^{\vee\vee}$ is a reflexive $\mu_H$-semistable sheaf and $Q$ is a torsion sheaf supported in dimension at most one. Applying Lemma A.3 to the exact sequence (40) shows that $E^{\vee\vee} = \mathcal{O}_Y^{\oplus 2}$ and $Q$ is a torsion sheaf of class $\mathrm{ch}(Q) = (0, 0, H^2, 0)$. Since $E \in \mathcal{K}u(Y)$ and $\mathrm{RHom}(\mathcal{O}_Y(1), E^{\vee\vee}) = 0$, we know that $H^0(Q(-1)) = 0$. Then the result follows from Lemma A.7. □

Recall that there is a natural double covering $\pi\colon Y \to \mathbb{P}^3$ (cf. Section 7.1).

**Lemma A.5.** *Let $Z \subset Y$ be a one-dimensional closed subscheme with $H.Z = 1$. If $Z$ is pure, then $Z$ is a line.*

**Proof.** Since $H.Z = 1$, we see $Z$ is irreducible since it is pure. Then $H.Z_{red} = 1$, which implies that $\ker(\mathcal{O}_Z \to \mathcal{O}_{Z_{red}})$ is zero-dimensional. But this is impossible since $\mathcal{O}_Z$ is pure. Hence $Z$ is integral, and $\pi(Z) \subset \mathbb{P}^3$ is also an integral subscheme of degree one, which is a line. Since $Z \subset \pi^{-1}(\pi(Z))$ is an irreducible component, $\pi^{-1}(\pi(Z))$ is reducible. Hence $\pi^{-1}(\pi(Z))$ is union of two lines on $Y$, which implies that $Z$ is a line. □

**Lemma A.6.** *Let $C \subset Y$ be a pure one-dimensional closed subscheme with $H.C = 2$ and $\chi(\mathcal{O}_C) = 0$. Then $C$ is irreducible and is the intersection of two hyperplane sections of $Y$. Moreover, $C = \pi^{-1}(\pi(C))$ and $\pi(C) \subset \mathbb{P}^3$ is a line.*

**Proof.** If $C$ is reducible, then from $H.C = 2$, each component is pure-dimensional with degree one, which is a line by Lemma A.5. Then these two components are either disjoint which implies $\chi(\mathcal{O}_C) = 2$, or intersect at a single point, which gives $\chi(\mathcal{O}_C) = 1$. Hence $C$ is irreducible.

If $H.C_{red} = 2$, then $C$ is reduced since $\mathcal{O}_C$ is pure. Then $\pi(C)$ is also integral. If the degree of $\pi(C)$ is two, then $C \cong \pi(C)$ which contradicts [42, Corollary 1.38] since $\chi(\mathcal{O}_C) = 0$. Thus $\pi(C)$ is a line, and $C \subset \pi^{-1}(\pi(C))$. Since $\pi^{-1}(\pi(C))$ is also a degree two curve of genus one, we have $C = \pi^{-1}(\pi(C))$.

If $H.C_{red} = 1$, then $C_{red} = l$ is a line, and we have an exact sequence $0 \to \mathcal{O}_l(-2) \to \mathcal{O}_C \to \mathcal{O}_l \to 0$. Thus $h^0(\mathcal{O}_C(1)) = 2$. Therefore, we have $h^0(\mathcal{I}_C(1)) \geq 2$ and $C$ is contained in two different hyperplane sections $S, S'$ of $Y$. This implies that $C \subset S \cap S'$. Since $S \cap S'$ is also a degree two curve of genus one, we have $C = S \cap S' = \pi^{-1}(l)$. □

**Lemma A.7.** *Let $Q$ be a coherent sheaf on $Y$ of class $\mathrm{ch}(Q) = (0, 0, H^2, 0)$ with $H^0(Q(-1)) = 0$ on $Y$. Then $Q$ is either*

(1) *an extension of structure sheaves of lines on $Y$,*
(2) *$Q = \theta_C(1)$, where $\theta_C$ is the theta characteristic of a smooth conic $C$ on $Y$, or*
(3) *$Q$ is a sheaf on a codimension two linear section $C$ of $Y$ given by*

$$0 \to \mathcal{O}_C \to Q \to R \to 0,$$

*where $R$ is a zero-dimensional sheaf on $C$ of length two.*

**Proof.** Since $\chi(Q) = 2$, we have $H^0(Q) \neq 0$. Let $s\colon \mathcal{O}_Y \to Q$ be a non-zero map. Then $\mathrm{im}(s) = \mathcal{O}_Z$, where $Z \subset Y$ is a subscheme. Since $H^0(Q(-1)) = 0$, we see $H^0(\mathcal{O}_Z(-1)) = 0$ and hence $Z$ is pure-dimensional. Note that if $H.Z_{red} = H.Z$, then the kernel of $\mathcal{O}_Z \to \mathcal{O}_{Z_{red}}$ is zero-dimensional, which implies $Z = Z_{red}$ by $H^0(\mathcal{O}_Z(-1)) = 0$. Let $R := \mathrm{coker}(s)$.

— Assume that $H.Z = 1$. Then by Lemma A.5, $Z$ is a line and hence $H^1(\mathcal{O}_Z(-1)) = 0$. Thus $\mathrm{ch}(R) = (0, 0, \frac{H^2}{2}, 0)$ and $H^0(R(-1)) = 0$. We claim that $R$ is also the structure sheaf of a line. Indeed, by $\chi(R) = 1$, we have a non-zero map $s'\colon \mathcal{O}_Y \to R$. By the same argument above, we see $H^0(\mathrm{im}(s')(-1)) = 0$ and hence $\mathrm{im}(s')$ is the structure sheaf of line by Lemma A.5. By the reason of Chern characters, we see $\mathrm{im}(s') = R$ and the result follows.

— Assume that $H.Z = 2$. First, we assume that $R = 0$, hence $\mathcal{O}_Z = Q$. If $H.Z_{red} = 1$, then $Z_{red}$ is a line by Lemma A.5. Thus $\ker(\mathcal{O}_Z \to \mathcal{O}_{Z_{red}})$ satisfies properties of $R$ in the first case. The same argument shows that $\ker(\mathcal{O}_Z \to \mathcal{O}_{Z_{red}})$ is also the structure sheaf of a line. If $H.Z_{red} = 2$, then the kernel of $\mathcal{O}_Z \to \mathcal{O}_{Z_{red}}$ is zero-dimensional, which implies $Z = Z_{red}$ by $H^0(\mathcal{O}_Z(-1)) = 0$. Note that $Z$ is reducible, otherwise we have $h^0(\mathcal{O}_Z) = 1$, which contradicts $\chi(\mathcal{O}_Z) = \chi(Q) = 2$. Hence by Lemma A.5, each of the irreducible components of $Z$ is a line. Since $\mathrm{ch}(\mathcal{O}_Z) = (0, 0, H^2, 0)$, we see $Z$ is an extension of structure sheaves of two lines.

 Now we assume that $R \neq 0$. The same argument as in [11, Lemma 3.3] shows that $Q$ is a $\mathcal{O}_Z$-module.

 — If $Z$ is reducible, then each component of $Z$ has degree one. Hence $H.Z_{red} = H.Z = 2$. This implies $Z = Z_{red}$ as above since $H^0(\mathcal{O}_Z(-1)) = 0$. By Lemma A.5, $Z$ is a union of two lines. And from $R \neq 0$, we see these two lines intersect with each other. In other words, $Z$ is a reducible conic. Now since $Z$ is a conic, the same argument as in [11, Lemma 3.3] shows that $Z$ is a smooth conic and $Q = \theta_Z(1)$.

 — If $Z$ is irreducible and $H.Z_{red} = 2$, then we also have $Z = Z_{red}$, which implies that $h^0(\mathcal{O}_Z) = 1$ and $\chi(\mathcal{O}_Z) \leq 1$. From [31, Lemma 4.3], we see $0 \leq \chi(\mathcal{O}_Z) \leq 1$. When $\chi(\mathcal{O}_Z) = 1$, $Z$ is also a conic, hence the same argument as in [11, Lemma 3.3] shows that $Z$ is a smooth conic and $Q = \theta_Z(1)$. When $\chi(\mathcal{O}_Z) = 0$, $Z$ is the intersection of two hyperplane sections by Lemma A.6 and hence $\mathrm{length}(R) = 2$.

 — If $Z$ is irreducible and $H.Z_{red} = 1$, then $Z_{red}$ is a line by Lemma A.5. Therefore, we have an exact sequence $0 \to \mathcal{O}_l(-n) \to \mathcal{O}_Z \to \mathcal{O}_l \to 0$, where $n \in \mathbb{Z}_{>0}$. In particular, we have $h^0(\mathcal{O}_Z) = 1$ which implies $\chi(\mathcal{O}_Z) \leq 1$. From [31, Lemma 4.3], we see $0 \leq \chi(\mathcal{O}_Z) \leq 1$. When $\chi(\mathcal{O}_Z) = 1$, $Z$ is a conic. By [11, Lemma 3.3], $Z$ is smooth and contradicts $H.Z_{red} = 1$. When $\chi(\mathcal{O}_Z) = 0$, we have $\mathrm{length}(R) = 2$ and the result follows. □

Now assume $E$ is a Gieseker-semistable reflexive sheaf of class $2\mathbf{v}$. It follows from [14, Proposition 2.6] that $E$ is a locally free sheaf and it is a slope stable locally free sheaf by Lemma A.2.

**Lemma A.8.** *Let $E \in M_Y(2, 0, 2)$ be a bundle with $E \in \mathcal{K}u(Y)$, then $E(1)$ is globally generated and it fits into the short exact sequence*

$$0 \to \mathcal{O}_Y(-H) \to E \to I_D(H) \to 0,$$

*where $D$ is a degree 4 smooth elliptic curve as the zero locus of a general section of $E(1)$.*

**Proof.** Note that $H^3(E(-2)) = H^0(E^\vee) = H^0(E) = 0$ since $E^\vee \cong E$. Then from $E \in \mathcal{K}u(Y)$, we see $H^i(E(1-i)) = 0$ for $i > 0$, thus $E(1)$ is globally generated by Castelnuovo–Mumford regularity. Then the zero locus of a generic section of $E(1)$ is smooth. The remaining statement follows from the Serre correspondence. □

On the other hand, the next proposition characterizes a semistable sheaf of rank two, $c_1 = 0, c_2 = 2, c_3 = 0$, which is not in the Kuznetsov component $\mathcal{K}u(Y)$.

**Proposition A.9.** *Let $E \in M_Y(2,0,2)$, then $E \notin \mathcal{K}u(Y)$ if and only if $E$ is locally free and fits into an exact sequence of the form* (39)*.*

**Proof.** By Lemma 3.1, we have $\mathrm{RHom}(\mathcal{O}_Y, E) = 0$. Note that $H^0(E(-1)) = H^3(E(-1)) = 0$ by Serre duality and stability. Thus from $\chi(E(-1)) = 0$, we see $E \notin \mathcal{K}u(Y)$ if and only if $H^1(E(-1)) = H^2(E(-1)) \neq 0$.

First we assume that $E$ fits into an exact sequence as above. Since $\mathcal{Q}_Y$ is a $\mu_H$-stable vector bundle by Lemma 3.3, it is clear that there is a non-zero morphism $E \to \mathcal{O}_Y(-1)[1]$, then $\mathrm{Hom}(\mathcal{O}_Y, E(-1)[2]) = \mathrm{Hom}(E, \mathcal{O}_Y(-1)[1]) \neq 0$ by Serre duality.

Now we assume that $H^1(E(-1)) \neq 0$. Applying $\mathrm{Hom}(-, E)$ to (9) and using $\mathrm{RHom}(\mathcal{O}_Y, E) = 0$, we have $\mathrm{Hom}(\mathcal{Q}_Y, E) = H^1(E(-1)) \neq 0$. Let $\pi \neq 0 \in \mathrm{Hom}(\mathcal{Q}_Y, E)$. We claim that $\pi$ is surjective and $\ker(\pi) \cong \mathcal{O}_Y(-H)$. Indeed, if $\mathrm{rk}(\mathrm{im}(\pi)) = 2$, then $\ker(\pi)$ is a reflexive torsion-free sheaf of rank one since $\mathcal{Q}_Y$ is locally free and $E$ is torsion-free. From the smoothness of $Y$, we know that $\ker(\pi)$ is a line bundle. By the $\mu_H$-semistability of $\mathcal{Q}_Y$ and $E$, we know that $c_1(\mathrm{im}(\pi)) = 0$, i.e. $c_1(\ker(\pi)) = -H$ and $\ker(\pi) = \mathcal{O}_Y(-H)$. Therefore, we only need to show that $\mathrm{rk}(\mathrm{im}(\pi)) \neq 1$.

To this end, we assume that $\mathrm{rk}(\mathrm{im}(\pi)) = 1$. Then by the $\mu_H$-semistability, we have $c_1(\mathrm{im}(\pi)) = 0$. Thus $\mathrm{ch}_{\leq 2}(\mathrm{im}(\pi)) = (1, 0, -\frac{a}{2}H^2)$ for $a \geq 1$. But we also know that Gieseker-stable implies 2-Gieseker-stable for $E$ by Lemma 3.2. Thus the only possible case is $a \geq 2$. Then $\mathrm{ch}_{\leq 2}(\ker(\pi)) = (2, -H, \frac{a-1}{2}H^2)$ with $a - 1 \geq 1$. But from the stability of $\mathcal{Q}_Y$, we know that $\ker(\pi)$ is also $\mu_H$-stable. This contradicts [27, Proposition 3.2]. Then the claim is proved.

The only part we remain to show is the local freeness of $E$. Assume that $E$ fits into (39). If $E$ is not reflexive, then as in Proposition A.4, we get $E^{\vee\vee} = \mathcal{O}_Y^{\oplus 2}$. However, using (39) we can compute that $\mathrm{Hom}(E, \mathcal{O}_Y) = 0$, which makes a contradiction. Thus $E$ is reflexive, and by $\mathrm{rk}(E) = 2$ and $c_3(E) = 0$, we see $E$ is locally free. □

*A.2. Singularities of moduli spaces*

In this section, we study singularities of stable moduli spaces $M_Y^s(2,0,2)$ and $\mathcal{M}_\sigma^s(\mathcal{K}u(Y), 2\mathbf{v})$.

**Lemma A.10.** *We have*

(1) $\mathrm{RHom}(\mathcal{O}_Y(1), \mathcal{Q}_Y) = \mathbb{C}[-1]$,
(2) $\mathrm{RHom}(\mathcal{Q}_Y, \mathcal{Q}_Y) = \mathbb{C}$, *and*
(3) $\mathrm{RHom}(\mathcal{O}_Y(-1), \mathcal{Q}_Y) = \mathbb{C}^6 \oplus \mathbb{C}[-1]$.

**Proof.** (1) follows from applying $\mathrm{Hom}(\mathcal{O}_Y(1), -)$ to (9). Note that $\mathrm{RHom}(\mathcal{O}_Y, \mathcal{Q}_Y) = 0$, then (2) follows from (1) and applying $\mathrm{Hom}(-, \mathcal{Q}_Y)$ to (9).

For (3), recall that $\pi_* \mathcal{O}_Y = \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-2)$. Since $\mathcal{Q}_Y = \pi^* \Omega_{\mathbb{P}^3}(1)$, we have $H^0(\mathcal{Q}_Y(1)) = H^0(\Omega_{\mathbb{P}^3}(2) \oplus \Omega_{\mathbb{P}^3})$. Thus $h^0(\mathcal{Q}_Y(1)) = 6$ by the standard result on $\mathbb{P}^3$. And by (9), we get $H^i(\mathcal{Q}_Y(1)) = 0$ for $i > 1$. Then the result follows from $\chi(\mathcal{Q}_Y(1)) = 5$. □

**Lemma A.11.** *We have* $\mathrm{RHom}(i^! \mathcal{Q}_Y, i^! \mathcal{Q}_Y) = \mathbb{C} \oplus \mathbb{C}^6[-1] \oplus \mathbb{C}[-2]$.

**Proof.** By the adjunction of $i$ and $i^!$, we have $\mathrm{RHom}(i^! \mathcal{Q}_Y, \mathcal{Q}_Y) = \mathrm{RHom}(i^! \mathcal{Q}_Y, i^! \mathcal{Q}_Y)$. Then the result follows from applying $\mathrm{Hom}(-, \mathcal{Q}_Y)$ to (12) and using Lemma A.10. □

**Lemma A.12.** *Let $E \in M_Y(2,0,2)$ and $E \notin \mathcal{K}u(Y)$, then* $\mathrm{RHom}(E, E) = \mathbb{C} \oplus \mathbb{C}^6[-1] \oplus \mathbb{C}[-2]$.

**Proof.** Since $E$ is stable, we have $\mathrm{Hom}(E,E)=\mathbb{C}$. And by stability we get $\mathrm{Ext}^3(E,E)=\mathrm{Hom}(E,E(-2))=0$. To prove the statement, we only need to show $\mathrm{ext}^2(E,E)=1$.

We compute $\mathrm{Ext}^2(E,E)$ via the standard spectral sequence (see e.g. [36, Lemma 2.27]) and (39). We have a spectral sequence with the first page

$$E_1^{p,q}=\begin{cases}\mathrm{Ext}^q(\mathcal{Q}_Y,\mathcal{O}_Y(-1)), & p=-1\\ \mathrm{Ext}^q(\mathcal{O}_Y(-1),\mathcal{O}_Y(-1))\bigoplus\mathrm{Ext}^q(\mathcal{Q}_Y,\mathcal{Q}_Y), & p=0\\ \mathrm{Ext}^q(\mathcal{O}_Y(-1),\mathcal{Q}_Y), & p=1\\ 0, & p\le -2, p\ge 2\end{cases}$$

and convergent to $\mathrm{Ext}^{p+q}(E,E)$. Then using Lemma A.10, we obtain $\mathrm{ext}^2(E,E)=1$ and the result follows.

**Remark A.13.** Denote by $M^{ni}$ the locus of Gieseker-semistable sheaves $E\in M_Y(2,0,2)$ but $E\notin\mathcal{K}u(Y)$. By Lemma A.12 the locus $M^{ni}$ is everywhere singular. But according to Lemma A.10 and (39), the reduction $M^{ni}_{red}$ of such locus is isomorphic to $\mathbb{P}\mathrm{Hom}(\mathcal{O}_Y(-1),\mathcal{Q}_Y)\cong\mathbb{P}^5$. In the following section A.3, we show it is contracted to a singular point in the Bridgeland moduli space $\mathcal{M}_\sigma(\mathcal{K}u(Y),2\mathbf{v})$ via projection functor $i^*$. □

*A.3. Bridgeland moduli space*

Finally, we study the relation between $M_Y(2,0,2)$ and $\mathcal{M}_\sigma(\mathcal{K}u(Y),2\mathbf{v})$.

**Lemma A.14.** *Let $E\in M_Y(2,0,2)$ such that $E\notin\mathcal{K}u(Y)$. Then $i^*E\cong i^!\mathcal{Q}_Y$.*

**Proof.** Note that $i^*\mathcal{O}_Y(-1)[1]\cong i^!\mathcal{Q}_Y$. Then applying $i^*$ to (39), we only need to show $i^*\mathcal{Q}_Y\cong 0$. By definition, we get an exact triangle

$$\mathcal{O}_Y(1)[-1]\xrightarrow{s}\mathcal{Q}_Y\to\mathbf{L}_{\mathcal{O}_Y(1)}\mathcal{Q}_Y,$$

where $s$ is the unique non-zero map in $\mathrm{Hom}(\mathcal{O}_Y(1)[-1],\mathcal{Q}_Y)$ up to scalar. We claim that the induced map

$$\mathbf{L}_{\mathcal{O}_Y}(s)\colon\mathbf{L}_{\mathcal{O}_Y}\mathcal{O}_Y(1)[-1]\to\mathbf{L}_{\mathcal{O}_Y}\mathcal{Q}_Y$$

is an isomorphism, which implies $i^*\mathcal{Q}_Y\cong 0$. Indeed, we have an exact triangle

$$\mathcal{O}_Y(1)[-1]\xrightarrow{s}\mathcal{Q}_Y\to\mathcal{O}_Y^{\oplus 4}$$

which comes from (9). Since $\mathbf{L}_{\mathcal{O}_Y}\mathcal{O}_Y\cong 0$, the claim follows. □

**Proposition A.15.** *Let $Y$ be a quartic double solid and $\sigma$ be a Serre-invariant stability condition on $\mathcal{K}u(Y)$. Then the projection functor $i^*$ induces a morphism*

$$p\colon M_Y(2,0,2)\twoheadrightarrow\mathcal{M}_\sigma(\mathcal{K}u(Y),2\mathbf{v})$$

*such that contracts $M^{ni}$ to a singular point represented by $i^!\mathcal{Q}_Y$, and is an isomorphism outside $M^{ni}$.*

**Proof.** Note that up to shift, all strictly $\sigma$-semistable objects are extensions of two ideal sheaves of lines, which are exactly all strictly Gieseker-semistable of class $2\mathbf{v}$ by Theorem A.1. Thus $i^*$ affects nothing on the strictly Gieseker-semistable locus. From Lemma 4.4, we also know that $i^!\mathcal{Q}_Y$ is $\sigma$-stable. Then the result follows from Theorem A.1, Lemma A.14 and Lemma A.11. □

**Remark A.16.** It looks plausible that for generic quartic double solids $Y$, the only singular point in $\mathcal{M}_\sigma(\mathcal{K}u(Y), 2\mathbf{v})$ would be the point $[i^!\mathcal{Q}_Y]$. As a result, up to composing with $\mathbf{O}$ and $[1]$, any exact equivalence $\Phi : \mathcal{K}u(Y) \simeq \mathcal{K}u(Y')$ would send $i^!\mathcal{Q}_Y$ to $i'^!\mathcal{Q}_{Y'}$, then by Theorem 6.2, we can get an alternative proof of categorical Torelli theorem for *generic* quartic double solids.

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